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A133335

FTD-ID(RS)T-0610-83

FOREIGN TECHNOLOGY DIVISION

METHODS OF THE ANALYSIS OF THE DISRUPTION OF TRACKING

by

G.V. Obrezkov, V.D. Razevig

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UNEDITED MACHINE TRANSLATION

FTD-ID(RS)T-0610-83 2 September 1983

MICROFICHE NR: FTD-83-C-001090

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English pages: 401

Source: Metody Analiza Sryva Slezheniya, "Sovetskoye Radio", Moscow, 1972, pp. 1-239

Country of origin: USSR
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PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

GRAPHICS DISCLAIMER

All figures, graphics, tables, equations, etc. merged into this translation were extracted from the best quality copy available.

METHODS OF THE ANALYSIS OF THE DISRUPTION OF TRACKING.

G. V. Obrezkov, V. D. Razevig.

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In the book is given the survey/coverage of the most important methods of the analysis of the disruption/separation of tracking in the locked followers of automatic radio equipment under the effect of fluctuating interferences. As examples is examined the phenomenon of the disruption/separation of tracking in the diagrams of the self-alignment of frequency and phase, in the systems of the automatic tracking of radar targets. The analytical methods of study, given in the book, rest in essence on the apparatus for Markov processes. Special attention is given to the analysis of the disruption/separation of tracking with the help of the analog and digital computers. Besides the direct application/appendix to the study of the disruption/separation of tracking the material can be useful, also, with the research of other nonlinear phenomena in radio engineering and the automation.

The book is intended for scientific workers and engineers, who carry out research and design of the radio engineering systems of automatic tracking.

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PREFACE.

The tendency to fully automate the work of radio sets and to maximally decrease the role of the man-operator led to the wide acceptance in radio engineering of followers. Specific for the work of the radio engineering systems of automatic tracking is the action of the fluctuating interferences, which are usually present in the receiving circuit. Beside the fact that the fluctuations worsen/impair the accuracy of the work of followers, appears the danger of the disturbance/breakdown of the very mode/conditions of tracking, i.e., disruption/separation. With this phenomenon it is necessary to be counted during the design of many radio engineering systems.

The methods of the analysis of the disruption/separation of tracking began be developed to intensely only in recent years. This is explained by the considerable mathematical difficulties, which appear during the solution of in principle nonlinear problems which include the analysis of disruption/separation. In connection with this the theory of the disruption/separation of tracking is at present presented, as a rule, only in the periodical articles. The

dispersion of information according to different methods of analysis creates known difficulties for the specialists, who carry out research and design of servo systems. In this book is undertaken the attempt to generalize and to systematize available material according to the analysis of the disruption/separation of tracking.

On the pages of the book the analysis of disruption/separation is conducted, as a rule, on the basis of the block diagram of device/equipment without the concrete definition of the functional designation/purpose of one or the other network elements.

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Therefore, it is possible to consider radio engineering devices/equipment different in the designation/purpose from the single systematic positions. However, in order to facilitate performance calculation of disruption/separation in the concrete/specific/actual device/equipment, in Chapter 1 are given the fundamental principles, which make it possible to determine the parameters of block diagrams for different systems of automatic tracking.

Analytical research of the disruption/separation of tracking in essence is based on the theory of Markov processes, the series/row of

information from which is presented in Chapter 2. Here considerable attention is given to practically important questions of the construction of Markov models for describing of servo systems and correct recording of boundary conditions for the multidimensional equations of Fokker - Planck and Pontriagin.

Material of the book is dedicated to the calculation of the probability of disrupting/separating the tracking for the assigned time interval (Chapter 3, 4, 6). In this case, as it seems to us, it was possible to consider the majority of the methods, known at present, which carry more or less general character.

In Chapter 5 is assembled the material according to the analysis of the less total characteristics of disruption/separation. They include, for example, mean time to the disruption/separation and critical power of fluctuations.

Widespread putting into engineering practice of the means of electronic computational engineering makes available research of complicated nonlinear regulating circuits whose analytical analysis to carry out difficultly. The questions, which relate to the numerical methods of the analysis of the disruption/separation of tracking, are examined in Chapter 6.

During the writing of the book the preference was given up not to strict mathematical proofs, but to the physical treatment of methods and phenomena. The presentation of material in the majority of the cases is illustrated by specific examples. Therefore the book can be available to readers having the information about the probability theory in the limits of the program of general technical VUZ [Institute of Higher Education].

The majority of the methods, examined in the book, is applicable not only for the analysis of the disruption/separation of tracking. The material of the book can be useful, for example, during the research of capture mode in the servo systems, during the analysis of some modes of operation of self-excited oscillators and during the calculation of the parameters of the ejections of random processes.

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The participation of the authors in the work on the book was expressed as follows: S 4.3, 6.3, 6.4 were written by V. D. Razevig, Chapter 2 and S 4.2 - by both authors together, remaining material was written by G. V. Obrezkov.

To the writing of the book in many respects contributed the scientific seminars and the consultations, conducted by Cand. the

physics and mathematics department. of sciences graduate student F. V. Shirokov and the Dr. of tech. sciences professor L. S. Gutkin. The authors express appreciation to them and to all participants in the seminars. The authors are grateful to all comrades, who read the manuscript, who took part in its discussion, and especially they wish to note the great work, carried out by official reviewers of the book by Prof. V. I. Tikhonov and by Prof. I. A. Bol'shakov. The authors express a deep appreciation for the constant attention to the work and friendly support to docent S. V. Pervachev, transactions and ideas of whom in many respects were used as basis for the writing of this book.

Devoting the book of memory of one of their teachers, V. L. Lebedev, the authors hope thus at least to partially express gratitude for that situation of friendly participation and benevolence of which it was accompanied work in its laboratory, and to note the large services of V. L. Lebedev in the development of the theory of statistical radio engineering.

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Chapter 1.

NONLINEAR SERVO SYSTEMS THEIR ELEMENTS.

In spite of the large diversity of the radio engineering systems of automatic tracking, frequently it is possible to manufacture single approach to their research. The significant role in this case plays the study of the work of followers on the basis of the analysis of their block diagrams. In this chapter are examined the methods of the composition of block diagrams and are investigated their characteristics for different systems of automatic tracking.

1.1. Block diagram of the system of tracking.

The majority of the radio engineering servo systems is constructed on the functional diagram, depicted in Fig. 1.1. Input signal $u_{вх}(\lambda, t)$, which carries information about the tracked parameter $\lambda(t)$, enters discriminator 1. At the output of discriminator as a result of the comparison of signals $u_{вх}(\lambda, t)$ and $u_{вых}(\hat{\lambda}, t)$ is formed stress/voltage $u_1(x, t)$, which depends on error $x(t)=\lambda(t)-\hat{\lambda}(t)$ of the disagreement/mismatch between the input (measured) parameter $\lambda(t)$ and

its estimation $\hat{\lambda}(t)$, which is formed as a result of the work of the ring of tracking.

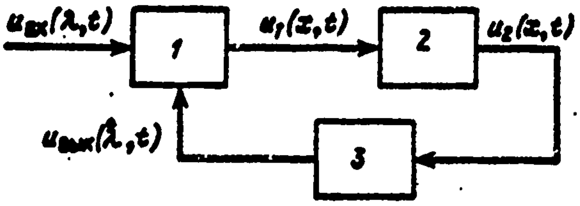

Fig. 1. 1. Typical functional diagram of servo system
1. discriminator; 2. filter; 3. control circuit

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Due to the presence in input signal $u_{вх}(\lambda, t)$ of interferences stress/voltage $u_1(x, t)$ fluctuates; therefore for increasing the accuracy of tracking into the system usually are introduced filtering cascades/stages 2. Stress/voltage $u_2(x, t)$ from the output of filter is supplied to the diagram of control of 3. The latter develops signal $u_{эвх}(\hat{\lambda}, t)$, which is modulated by estimation $\hat{\lambda}(t)$ in the same way as input signal $u_{вх}(\lambda, t)$ by the parameter $\lambda(t)$. Depending on the designation/purpose of diagram from it are removed/taken either stress/voltage $u_{эвх}(\hat{\lambda}, t)$, or stress/voltage from other points of diagram, proportional to certain function of estimation $\hat{\lambda}(t)$ (for example, by its derivative).

Let us pause at the short characteristic of the elements/cells of the functional diagram, depicted in Fig. 1.1.

Discriminator. Device/equipment is in principle nonlinear, which is necessary for the isolation/liberation (demodulation) of the signal, proportional to the mismatch error of the parameters $\lambda(t)$ and $\hat{\lambda}(t)$. The latter, as a rule, are not additive with respect to their

carriers - stresses/voltages $u_{\text{вх}}(\lambda, t)$ and $u_{\text{вых}}(\hat{\lambda}, t)$. However, for the parameters λ and $\hat{\lambda}$ at the sufficiently low value of disagreement/mismatch $x=\lambda-\hat{\lambda}$ discriminator can be considered linear device/equipment. Here is outlined analogy with the amplitude detector, nonlinear according to the principle of its operation, but linear for the signal amplitude envelope.

Subsequently it is convenient to be abstracted from the method of modulation of input and output signals by the parameters λ and $\hat{\lambda}$ and to use with the block diagram (Fig. 1.2) of the device/equipment of tracking the parameter $\lambda(t)$.

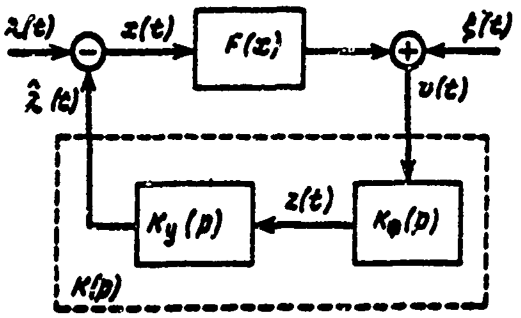

Fig. 1.2. Standard block diagram of the tracking device.

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Discriminator Fig. 1.2 presents by the upper part of the drawing. In this case is separately isolated the subtractor, which develops mismatch error $x(t)=\lambda(t)-\hat{\lambda}(t)$ between the measured parameter $\lambda(t)$ and its estimation $\hat{\lambda}(t)$.

The dependence, which connects the mathematical expectation of output potential of real discriminator with disagreement/mismatch x, in Fig. 1.2 is designated by F(x). This dependence is conventionally designated as discriminatory characteristic. Characteristics F(x) of some concrete/specific/actual types of discriminators are investigated in § 1.2. Range of values x, output beyond limits of which leads to the disruption/separation of tracking, let us name the aperture of discriminatory characteristic, or, shorter, by the aperture of discriminator.

Fluctuating voltage component at the output of discriminator in the block diagram is considered by the introduction of random process $\xi(t)$ with a spectral density of $N_\omega(x)$. Here and everywhere subsequently by spectrum is understood the following Fourier transform above the correlation function $r(\tau)$:

$$N_\omega = 2\int_{-\infty}^{\infty} r(\tau)\, e^{-j\omega\tau} d\tau = 4\int_{0}^{\infty} r(\tau)\cos\omega\tau\, d\tau. \qquad (1.1)$$

The passband of the radio engineering servo systems licks usually in the limits from zero to ones, and rarer - tens of Hertz. In this frequency band the dependence of spectral density $N_\omega(x)$ on the frequency ω is expressed weakly; therefore frequently assume/set $N_\omega(x) = N_0(x)$, considering noise $\xi(t)$ white. The dependence of spectral density $N_0(x)$ on the mismatch error x occurs in many types of discriminators and is called fluctuating characteristic. The standard fluctuating characteristics of real discriminators are examined in S 1.2.

For evaluating the quality of the work of discriminators frequently is used the coefficient, that characterizes signal-to-noise ratio at the output:

$$\varkappa = k\frac{U_m^2}{N_0(0)},$$

where U_m— the maximum stress/voltage, removed from the output of discriminator; k - the dimension factor of proportionality.

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Performance calculation F(x) and $N_o(x)$ represents independent, now and then very complex problem for each concrete/specific/actual type of discriminators. Using subsequently only with the block diagrams of followers, we consider characteristics F(x) and $N_o(x)$ known. To their calculation is dedicated the very vast literature whose short survey/coverage is given in the following paragraph.

In the majority of the practical cases discriminator it suffices to consider nonlinear inertia-free component/link. However, there are the situations, when the inertness of discriminator cannot be disregarded/neglected in comparison with the inertness of the remaining part of the diagram of tracking. In this case frequently it is possible to approximately represent the block diagram of discriminator in the form of series connection of the inertia-free block of nonlinearity with characteristic F(x) and linear inertia element/cell. This leads as a result to an increase in the dimensionality of the differential equation, which describes the behavior of the system of tracking.

Filtering cascades/stages. Since useful output potential of discriminator is a slowly varying function of time, then as the filters in the real systems usually are used low-pass filters. The widest use received the following types of filters: integrating, with the operational gear ratio/transmission factor $K_\Phi(p)=\frac{1}{1+pT}$ $\left(p=\frac{d}{dt}\right.$ — differential operator); proportional-integrating-$K_\Phi(p)=\frac{1+pT_1}{1+pT}$; active integrating-$K_\Phi(p)=\frac{K_0(1+pT)}{p}$ etc. The filtering cascades/stages, as a rule, are linear and as their complete characteristic serves operational gear ratio/transmission factor $K_\Phi(p)$.

Diagram of control. As has already been mentioned, its designation/purpose is reduced to modulation of stress/voltage $u_{вых}(\lambda, t)$ by the estimation of the parameter $\hat{\lambda}(t)$. During the analysis of servo system within the framework of its block diagram the method of modulation does not play role; therefore as the fundamental characteristic of the diagram of control serves dependence $\hat{\lambda}=\hat{\lambda}(z)$, where z(t) - the stress/voltage, removed from the output of filter.

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In the majority of the cases the characteristic $\hat{\lambda}(z)$ is linear in the limits, necessary for retaining/preserving/maintaining the mode/conditions of tracking. Feast this the diagram of control is

characterized only by conversion conductance $K_y = d\hat{\lambda}/dz$.

Sometimes the inertness of the diagram of control is commensurated with the inertness of filter 2 (see Fig. 1.1). For example, if in the system with the phase discriminator as the control device is used reactance tube, then this diagram of control is simultaneously ideal integrator with the operational gear ratio/transmission factor $K_y(p) = K_y/p$.

If the feedback loop of control system consists only of linear elements/cells, then it are conveniently characterized by the operational gear ratio/transmission factor, which encompasses the gear ratios/transmission factors of filter and diagram of control $K(p) = K_\phi(p)\,K_y(p)$.

Differential equation. With the help of the block diagram it is easy to register the differential equation, which describes the behavior of the system of tracking. Thus, on the block diagram, depicted in Fig. 1.2, for the following error x(t) we have

$$x(t) = \lambda(t) - K(p)\,[F(x) + \xi(t)]. \qquad (1.2)$$

Operational equation (1.2) is stochastic, since into it enters random function ξ(t). Revealing in each specific case the content of operator K(p), on basis (1.2) we obtain the differential equation of the analyzed servo system. For example, if the feedback loop of

system consists of the diagram of control, which is simultaneously integrator, so that $K_y(p) = K_y/p$, and the filter of lower first-order frequencies with the gear ratio/transmission factor $K_\phi(p)=K_\phi/(1+pT)$, then $K(p)=\frac{K}{p(1+pT)}$, where $K=K_\phi K_y$.

The behavior of this system is described stochastic differential equation of the second order:

$$T\frac{d^2x}{dt^2}+\frac{dx}{dt}+KF(x)=T\frac{d^2\lambda}{dt^2}+\frac{d\lambda}{dt}-K\xi(t).$$

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1.2. Characteristics of the most widely used discriminators.

The analysis of the disruption/separation of tracking cannot be produced without the knowledge of the characteristics of discriminatory device/equipment. The most important characteristics are dependences on disagreement/mismatch x of the constant component of F(x) and spectral density $N_\bullet(x)$ of process at the output of discriminator. The calculation of these characteristics is in the general case complicated and labor-consuming task, since it is necessary to consider the passage of signal and interference not only through the discriminator, which is nonlinear device/equipment, but also through entire circuit of receiver, which also contains in a number of cases substantially nonlinear components/links. In this

paragraph is given the short survey/coverage of the results available in the literature according to the analysis of the most important types of discriminators.

1. Temporary/time discriminators.

For the temporary/time discrimination of pulse video signal the widest use received the diagram, depicted in Fig. 1.3 [92]. The input voltage, which is the envelope of the mixture of noise and periodic pulse signal, enters the cascades/stages of coincidence KS_1 and KS_2. In these cascades/stages by gates/strobes U_{c1} and U_{c2} from the input voltage are cut out the impulses/momenta/pulses by duration T, shifted relative to each other the interval of time τ.

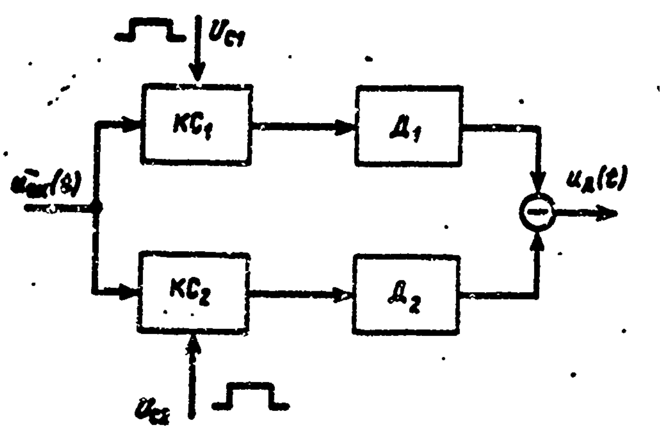

Fig. 1.3. The functional diagram of the temporary/time discriminator: KS - cascade/stage of coincidence; Д - detector.

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Impulses/momenta/pulses are supplied to the detectors D_1 and D_2, the results of detection are subtracted and are formed output stress/voltage $u_x(t)$ of temporary/time discriminator, depending on disagreement/mismatch x between the center of signal and the axis of the symmetry of gates/strobes.

To the analysis of different diagrams of temporary/time discriminators are dedicated works [91-93, 95, 97] and series/row of others (more complete bibliography on this question is given in [92]).

Is distinguished the work of temporary/time discriminator with the jettisoning and without jettisoning of stress/voltage on the detectors D_1 and D_2 before the arrival of next impulse/momentum/pulse. To the evaluation of the effect of jettisoning stress/voltage on the characteristics of discriminator is dedicated work [97]. As in it it is shown, the temporary/time discriminator without jettisoning of stress/voltage possesses the further filtering properties the large, is the more the time constant T_p of the discharge circuit of detector. Therefore in the diagram without the jettisoning in comparison with the discriminator with the jettisoning with increase T_p is reduced the dispersion of output

stress/voltage. However, both diagrams ensure at the output of discriminator virtually identical relation signal/noise.

With not too small a time constant of charging circuits of detectors, output potential of discriminator with the jettisoning after the next operation of selection can be represented in the form

$$u_{\text{д}} = k\int_{t_0}^{t_0+T} u(t)\,dt - k\int_{t_0+\tau}^{t_0+\tau+T} u(t)\,dt,$$

where t_0, $t_0+\tau$ - respectively the beginning of the first and second selecting impulses/momenta/pulses; T - duration of one selecting pulse; k - proportionality factor. Taking into account that in the pauses between the signal pulses occurs the discharge of the capacities/capacitances of detectors, for calculating the discriminatory characteristic it is possible to use the relationship/ratio

$$F(x) = \bar{u}_{\text{д}} = k\left[\int_{t_0}^{t_0+T} \overline{u(t)}\,dt - \int_{t_0+\tau}^{t_0+\tau+T} \overline{u(t)}\,dt\right] e^{-\alpha t},$$

where t - repetition period of the signal pulses; $\alpha = 1/T_p$, T_p — the time constant of the discharge circuit of detector.

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In the given relationship/ratio input voltage u(t) should be considered as the function of disagreement/mismatch x between signal

and axis of the symmetry of gates/strobes. Strictly speaking, characteristic F(x) is periodic function x with the repetition period l, equal to the repetition period of the signal pulses. However, with the large mark-space ratios ($l/\tau_0 \gg 1$) periodicity F(x) it is possible not to consider.

Most frequently the gates/strobes of temporary/time discriminator place directly one after another, so that τ=T. In this case, if signal is approximated by square pulse with duration T_0, the characteristic of discriminator F(x) depending on the duration of one gate/strobe T takes the form, shown in Fig. 1.4. The slope/transconductance of discriminatory characteristic in the region of the small disagreements/mismatches x~0 is maximum, if the duration of gates/strobes is not less than the duration of signal ($T \geqslant T_0$). In the mode/conditions of tracking usually use the gates/strobes, equal in the duration to signal. This ensures the best signal-to-noise ratio at the output of discriminator. If signal functions in the mixture with the noise, then during the calculation of discriminatory characteristic should be considered the effect of suppression of signal in the detector of the radio pulses of receiver [92].

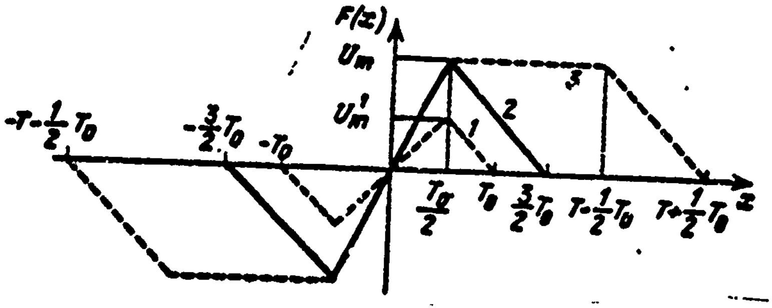

Fig. 1.4. Characteristics of the temporary/time discriminator: 1) $T=1/2\ T_0$; 2) $T=T_0$; 3) $T>T_0$.

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As a result for a maximum increase in the constant component of output potential of discriminator, which occurs during disagreement/mismatch $x=T_0/2$, it is possible to obtain the following expression:

$$U_m = K\sigma_0\sqrt{\frac{\pi}{2}}e^{-q^2}\left\langle e^{-q^2/2}\left\{I_0\left(\frac{q^2}{2}\right)+ +q^2\left[I_0\left(\frac{q^2}{2}\right)+I_1\left(\frac{q^2}{2}\right)\right]\right\}-1\right\rangle, \quad (1.3)$$

where $q=U_{c0}/\sqrt{2}\sigma_0$; U_{c0}, σ_0^2 — respectively signal amplitude and the dispersion of noise, led to the entrance of linear receiver; K - factor of amplification of receiver, switching on the cascade/stage of coincidence and detector; I_0, I_1 - modified functions of Bessel of the first order of zero and first order respectively.

As it was noted, with the work of temporary/time discriminator with the jettisoning output stress/voltage is (Fig. 1.5) the sequence of exponential impulses/momenta/pulses with the duration t, equal to the repetition period of the signal pulses. The amplitude of pulses U is by chance with dispersion $\sigma_U^2(x)$, which depends and the general case on disagreement/mismatch x. The spectrum of this stress/voltage takes the form

$$N_\omega(x)=\frac{2\sigma_U^2(x)}{(\alpha^2+\omega^2)t}[1+e^{-2\alpha t}-2e^{-\alpha t}\cos\omega t].$$

In the region of lower frequencies $\omega\sim0$ we have

$$N_0(x)=\frac{2\sigma_U^2(x)(1-e^{-\alpha t})^2}{\alpha^2 t}. \qquad (1.4)$$

To the determination of dispersion $\sigma_U^2(x)$ is dedicated, in particular, work [92]. In Fig. 1.6 according to the results of this work are constructed graphs $\sigma_U^2(x)$ in different ratios q of signal to the noise at the entrance of linear receiver.

During the calculation of curves it was assumed that the signal pulse has a rectangular form and a duration, the equal width of one gate/strobe ($T_0=T$). Furthermore, it was considered that frequency receiver response has Gaussian form with a bandwidth of $\Delta f=1/T_0$.

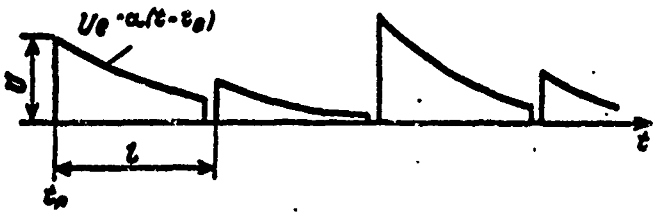

Fig. 1.5 output potential of discriminator with the jettisoning.

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As can be seen from Fig. 1.6, on a small level of signal ($q \leq 0{,}5$) the nonuniformity of fluctuating characteristic can be disregarded/neglected, assuming/setting

$$N_0(x) \approx N_0(0) = \frac{2\sigma_U^2(0)\,(1 - e^{-\alpha l})^2}{\alpha^2 l}.$$

For dispersion $\sigma_U^2(0)$ in work [92] is obtained the following expression, valid when $T \geq T_0$:

$$\sigma_U^2(0) = \frac{\pi}{2}\,\sigma_0^2\Big\{b_1\left(\frac{T_0}{2}\right)^2\left[\tilde{\gamma}_2^2\left(\frac{T_0}{2}\right) - \tilde{\gamma}_2^2(T_0)\right] +$$
$$+ b_2 T^2\,[\tilde{\gamma}_1^2(T) - \tilde{\gamma}_1^2(2T)] + \frac{1}{2}(1 - b_2)\times$$
$$\times\left(T - \frac{T_0}{2}\right)^2 \tilde{\gamma}_1^2\left(T - \frac{T_0}{2}\right)\Big\},$$

where

$$b_1 = 2q^2 e^{-q^2}\left[I_0\left(\frac{q^2}{2}\right) + I_1\left(\frac{q^2}{2}\right)\right]^2;$$
$$b_2 = e^{-q^2/2}\left[I_0^2\left(\frac{q^2}{2}\right) + I_1^2\left(\frac{q^2}{2}\right)\right];$$

$\hat{\gamma}_1$, $\hat{\gamma}_2$ - coefficients of the averaging of fluctuations at the output of the detector of radio pulses.

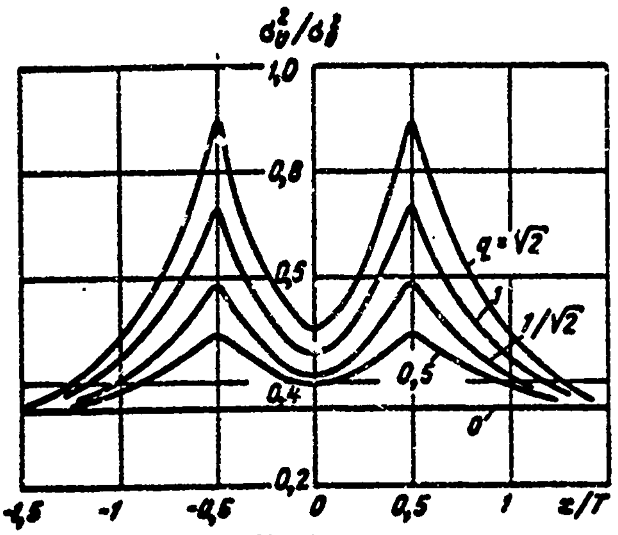

Fig. 1.6. Fluctuating characteristics of temporary/time discriminator.

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If the frequency receiver response has a form of gaussian curve, then

$$\widetilde{\gamma}_1^2(\tau)=\frac{1}{\sqrt{2}\,\Delta f\tau}\,\Phi(\sqrt{2\pi}\Delta f\tau)-\frac{1}{2\pi(\Delta f\tau)^2}\left[1-e^{-2\pi(\Delta f\tau)^2}\right],$$

$$\widetilde{\gamma}_2^2(\tau)=\frac{1}{\Delta f\tau}\,\Phi(\sqrt{\pi}\,\Delta f\tau)-\frac{1}{\pi(\Delta f\tau)^2}\left[1-e^{-\pi(\Delta f\tau)^2}\right],$$

where $\Phi(z)$ - the probability integral, equal to

$$\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx. \qquad (1.5)$$

The case when strobing/gating is produced by spike pulses ($T_0 \ll T$), spread up to the distance $\tau \sim T_0$, and detectors D_1 and D_2 are peak, it is examined in works [95, 97]. Discriminators of such type

are used in the practice considerably more thinly.

Sometimes the receiving circuit, which precedes temporary/time discriminator, contains the series/row of substantially nonlinear cascades/stages. To nom can relate the limiters, cascades/stages with the logarithmic amplitude characteristics, etc. Performance calculation of discriminators in this case substantially is complicated. Some results of performance calculation of discriminators in these cases are given in works [94, 96].

2. Phase discriminators.

Phase discriminators [91, 98, 99] extensively are used in many radio engineering devices/equipment. With their aid is realized, for example, phase tracking and frequency of received signal. They frequently are used in the devices/equipment of information recovery about angular target position in radars, etc..

The widest use obtained two types of phase discriminators - balance and commutation [91]. The diagram of balance discriminator is depicted in Fig. 1.7. Let us consider its work under the effect at the input of monochromatic signal $u_{вх}(t) = U_0 \sin(\omega t + \varphi)$, where φ - phase displacement of input signal relative to supporting/reference $u_{оп}(t) = U_0 \sin \omega t$.

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If amplitude detectors D_1 and D_2 are linear, then discriminatory characteristic can be represented [91] in the form

$$F(\varphi)=K_{\text{д}}U_m(\sqrt{1+h^2+2h\cos\varphi}-\sqrt{1+h^2-2h\cos\varphi}),$$

where $h=U_{\text{оп}}/U_m$ — the ratio of the amplitudes of reference and input signals on secondary windings of transformers; $K_{\text{д}}$ — the gear ratio/transmission factor of detectors D_1 and D_2.

Standardized/normalized discriminatory characteristics $f(\varphi)=F(\varphi)/2K_{\text{д}}U_m$ for the different values of coefficient of h are given in Fig. 1.8. With h=1 the discriminatory characteristic of phase discriminator has a form, close to the triangular. With the large amplitudes of reference voltage $U_{\text{оп}}\gg U_m$, which usually occurs in the real devices/equipment, discriminatory characteristic takes the form

$$F(\varphi)=2K_{\text{д}}U_m\cos\varphi. \qquad (1.6)$$

The analysis of the work of balance phase discriminator under the effect at its input of the mixture of signal and noise is given in work [98].

For the approximate computations of the characteristics of phase discriminator in the sufficiently large ratios of the amplitudes of

supporting/reference and input of signals ($h \geq 5$) phase discriminator can be replaced with the multiplier, which realizes the operation

$$u_{вых}(t) = k u_{вх}(t) u_{оп}(t).$$

In this case discriminatory characteristic is determined by expression (1.6), and the spectrum of the output stress/voltage in the region of lower frequencies coincides with the spectrum of input voltage near the frequency of reference oscillator.

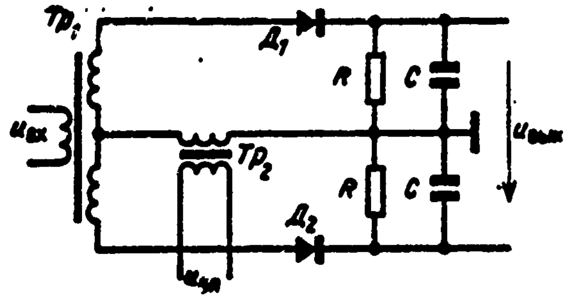

Fig. 1.7. Diagram of balance phase discriminator.

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Let us note that if in the diagram of balance discriminator are used detectors with the quadratic volt-ampere characteristics, then with the ideal symmetry of diagram the spectrum of input voltage is transferred to the zero frequency without the distortions with any amplitudes of reference signal.

Wide acceptance in the practice, especially with the work at the low frequencies, received commutation type phase discriminators (Fig. 1.9).

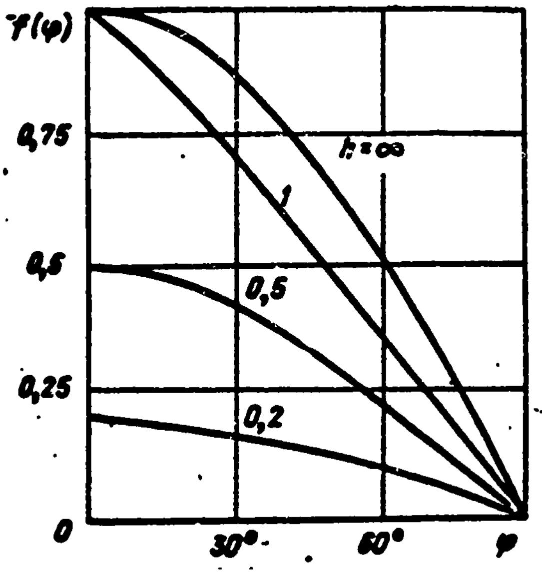

Fig. 1.8.

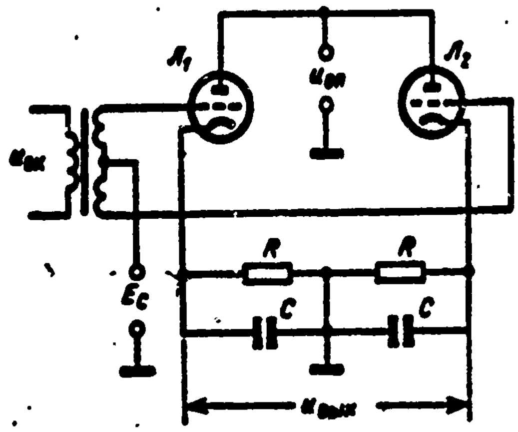

Fig. 1.9.

Fig. 1.8. Discriminatory characteristics of phase discriminator.

Fig. 1.9. Commutation- type phase discriminator.

Page 19.

Reference voltage $u_{оп}(t)$ in the detectors of such type usually has a form of meander with period $T=2\pi/\omega$, where ω - carrier frequency of input signal $u_{вх}(t)$. The constant component of stress/voltage $u_{вых}(t)$ on the total cathode load is determined by phase displacement φ between input and reference voltages [91]:

$$\bar{u}_{вых}=F(\varphi)=K_{фд}U_m\cos\varphi,$$

where $K_{\phi x}$ — the gear ratio/transmission factor of phase discriminator, equal to

$$K_{\phi x}=\left(\frac{du_{\text{вых}}}{dU_m}\right)_{\varphi=0}=\frac{2SR}{\pi\left(1+\frac{R}{2R_i}\right)};$$

S, R_i — respectively slope/transconductance and anode resistance.

Noise effect on commutation type phase discriminator is examined in work [99], where in particular, is found the expression of the spectral density of the output stress/voltage

$$N_{\phi}=\frac{8K_0^2}{\pi^2}F_{\phi}^2(\omega)\sum_{l=-\infty}^{\infty}\frac{N_{\text{вх}}[\omega-(2l+1)\omega_0]}{(2l+1)^2}, \qquad (1.7)$$

where $K_0=S/(S+1/R_i+1/R)$ — the gear ratio/transmission factor of cathode follower; $N_{\text{вх}}(\omega)$ — the spectral density of the input voltage, which is the additive mixture of signal and noise; $F_{\phi}^2(\omega)$ — the frequency characteristic of low-pass filter at the output of phase discriminator.

The spectrum of the output stress/voltage in the region of lower frequencies is determined by the member of sum (1.7), which corresponds to $l=-1$:

$$N_{\phi}=\frac{8K_0^2}{\pi^2}F_{\phi}^2(\omega)\,N_{\text{вх}}(\omega+\omega_0),$$

i.e. commutation phase discriminator similar to ideal multiplier realizes a transfer of the frequency of the input signal into the region of lower frequencies without the distortion of the form of the spectrum.

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Usually as the low-pass filter is used the integrating RC network (see Fig. 1.9). In this case

$$F_{\phi}^{2}(\omega)=\frac{1}{1+\omega^{2}T_{\phi}^{2}},\quad T_{\phi}=RC.$$

3. Frequency discriminators [1].

FOOTNOTE [1]. The material of this section is written together with Yu. A. Yevsikov. ENDFOOTNOTE.

Frequency discriminators are the devices/equipment, which convert frequency entering the stress/voltage. The output stress/voltage of discriminator $u_{д}(t)$ is obtained as a result of the comparison of frequency of input ω with certain standard frequency ω_{0}, for example by the resonance frequency of duct/contour or system of ducts/contours.

Among the works, dedicated to research of frequency discriminators during the combined action of signal and noise, one should note [1, 11, 100-105]. In the practice the widest use obtained

two types of the discriminators: discriminator on detuned circuits [91] (Fig. 1.10a) and discriminator with the duct/contour and phase inverter [1] (Fig. 1.10b). Both discriminators have the accuracy close to the optimum of the measurement of signal frequency with fluctuating interference [1].

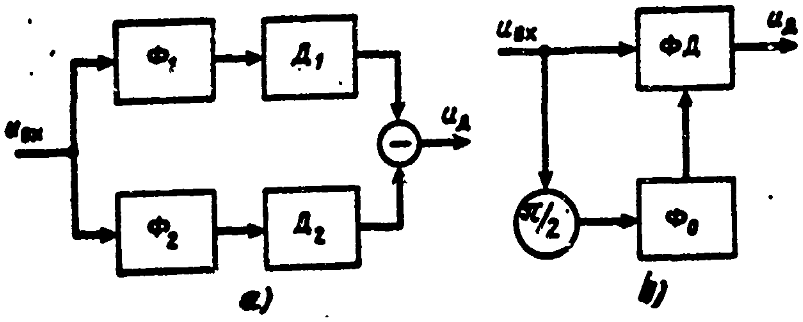

Fig. 1.10. Diagrams of the frequency discriminators: a) on the detuned circuits; b) with the duct/contour and the phase inverter. Ф - filter; Д - amplitude detector; FD - phase discriminator.

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In work [100] are obtained the discriminatory characteristics F(x) of frequency discriminator with the detuned circuits under the effect at the entrance of monochromatic signal $u_{вх}(t) = U_m \cos\omega_c t$ and normal broadband noise. In the case of the complete symmetry of the arms of discriminator and when the filters $Ф_1$ and $Ф_2$ are single oscillatory circuits, and amplitude detectors $Д_1$ and $Д_2$ are linear, characteristic F(x) is determined by the dependence

$$F(x) = \frac{U_m K}{q_0} \frac{\sqrt{\pi}}{2} [B(g_1) - B(g_2)], \qquad (1.8)$$

where K - gear ratio/transmission factor of one arm at the resonance frequency (taking into account the gear ratio/transmission factor of amplitude detector); $q_0 = U_{m0}/\sqrt{2}\sigma$, U_{m0} — maximum signal amplitude at the output of one duct/contour (with the coincidence of signal frequency

with the resonance frequency of duct/contour); σ - efficient noise voltage in the band of one duct/contour;

$$B(z)=e^{-z^2/2}\left\{I_0\left(\frac{z^2}{2}\right)+z^2\left[I_0\left(\frac{z^2}{2}\right)+I_1\left(\frac{z^2}{2}\right)\right]\right\};$$

$$g_1=\frac{q_0}{\sqrt{1+(x-x_0)^2}},\quad g_2=\frac{q_0}{\sqrt{1-(x-x_0)^2}};$$

$$x_0=\frac{\omega_1-\omega_2}{2\alpha\omega_0}$$

- dimensionless detuning of the resonance frequencies of the ducts/contours;

$$x=\frac{\omega_c-\omega_0}{\alpha\omega_0}$$

- generalized detuning of signal; α - attenuation factor of ducts/contours.

If is permitted a 20-30-percent error in the definition of characteristic F(x), then function B(z) can be calculated according to the approximation formula

$$B(z)\approx\frac{1}{z}\sqrt{1+\frac{0,7}{z}}.$$

In Fig. 1.11 are constructed standardized/normalized discriminatory characteristics $f(x)=F(x)/U_mK$, calculated according to formula (1.8) with x_0=1.5.

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The calculation of spectral density $N_\omega(x)$ of fluctuations at the output of discriminator with the detuned circuits is produced in work

[101] with the same assumptions as in [100]. During the analysis was used the approximation of the distribution of the signal amplitude envelope and noise by law of Nakagama. As a result was obtained the formula for calculating the spectral density $N_{*}^{'}(x)$ with the arbitrary detuning x and x_{*}.

For the case when signal can be represented by narrow-band normal random process, the calculation of spectral density $N_{*}^{'}(x)$ is carried out in work [102]. The results of this work are generalized in [104] in the case of nonuniform interference spectrum at the entrance of discriminator.

To the study of the passage of the fluctuating (in particular, harmonic) signal and interference with the arbitrary energy spectrum through the discriminators (see Fig. 1.10a, b) is dedicated work [105].

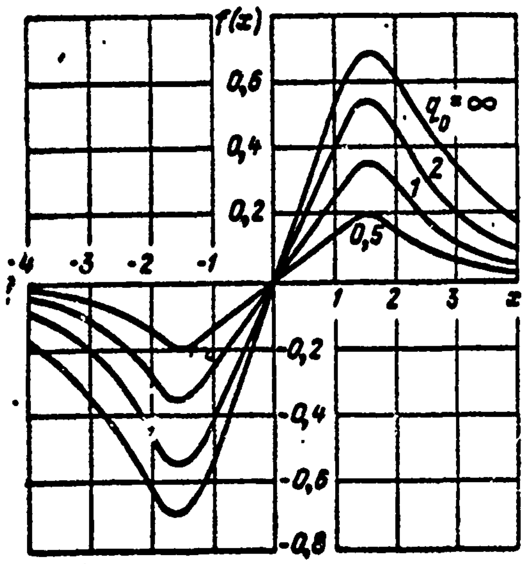

Fig. 1.11. Characteristics of frequency discriminator on the detuned circuits.

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It is assumed in it that the amplitude detectors of discriminator with the detuned circuits have square-law characteristics $u=K_{д}U_{m}^{2}$, where u - instantaneous output potential of detector, U_m— signal amplitude at the entrance, and the phase discriminator of discriminator with the phase inverter is ideal multiplier with the gear ratio/transmission factor $K_{фд}$. Then the mathematical expectation of the output stress/voltage of discriminators of both types is determined by the expression

$$F(\Omega_c, \Omega_n) = \frac{\mu_1}{2\pi} \int_{-\infty}^{\infty} \tilde{N}(\xi)\psi(\xi)\,d\xi, \qquad (1.9)$$

where $\tilde{N}(\Omega)=\tilde{N}_c(\Omega,\Omega_c)+\tilde{N}_n(\Omega,\Omega_n)=N(\Omega+\omega_0)$ — displaced into the region of lower frequencies the total energy spectrum of signal and interference by the entrance of discriminator; $\Omega=\omega-\omega_0$ - deviation of the current frequency ω from standard value ω_0; $\Omega_{c(n)}=\omega_{c(n)}-\omega_0$ — the divergence of the medium frequency of the spectrum of signal (interterence) from the frequency ω_0; $\psi(\Omega)$ - standardized static characteristic. Function $\psi(\Omega)$ and coefficient μ_1 depend on the type of discriminator. For the discriminator on the detuned circuits

$$\psi(\Omega)=k_1^2(\Omega)-k_2^2(\Omega),\quad \mu_1=K_A,$$

for the discriminator with the phase inverter

$$\psi(\Omega)=\operatorname{Im}\tilde{K}_0(j\Omega),\ \mu_1=K_{\phi n},$$

where $k_i(\Omega)=|\tilde{K}_i(j\Omega)|$; $\tilde{K}_i(j\Omega)=K_i(j\Omega+j\omega_0)$ — displaced complex gear ratios/transmission factors of filters Φ_i, which form part of the discriminators (see Fig. 1.10).

The energy spectrum of processes at the outputs of discriminators of both types is determined by the expression

$$N_n(\Omega_c,\Omega_n)=\frac{\mu_2}{\pi}\int_{-\infty}^{\infty}\tilde{N}(\xi)\tilde{N}(\xi+\Omega)\chi(\xi,\Omega)\,d\xi,\quad (1.10)$$

where for the discriminator on the detuned circuits

$$\mu_2 = K_x^2,$$

$$\chi(\xi, \Omega) = \overline{K}_1^2(\xi)\overline{K}_1^2(\xi+\Omega) + \overline{K}_2^2(\xi)\overline{K}_2^2(\xi+\Omega) -$$

$$-2\mathrm{Re}[\dot{K}_1(j\xi)\dot{K}^*_1(j\xi+j\Omega)\dot{K}^*_2(j\xi)\dot{K}_2(j\xi+j\Omega)],$$

for the discriminator with the phase inverter

$$\mu_2 = \frac{K_{\phi x}^2}{4},$$

$$\chi(\xi, \Omega) = \overline{K}_0^2(\xi) + \overline{K}_0^2(\xi+\Omega) - 2\mathrm{Re}[\dot{K}_0(j\xi)\dot{K}_0(j\xi+j\Omega)].$$

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Here K*(jω) - the function, complex conjugated with K(jω).

From expression (1.10) it follows that the spectral density in the frequency region, close to the zero, is equal to

$$N_0(\Omega_0, \Omega_n) = \frac{\mu_1^2}{\pi}\int_{-\infty}^{\infty}\overline{N}^2(\xi)\psi^2(\xi)\,d\xi. \qquad (1.11)$$

Relationships, atios 10), (1.11) are valid in the case of harmonic signal, if we assume

$$\overline{N}_e(\Omega, \Omega_0) = \pi U_m^2\,\delta(\Omega - \Omega_0).$$

As an example let us give expression for the standardized/normalized spectral density of process at the output of frequency discriminator with the duct/contour and the phase inverter, obtained on the assumption that at the entrance of discriminator functions the harmonic signal and white noise, that passed through the amplifier with amplitude-frequency characteristic:

$$\overline{K}_0(\Omega) = e^{-\Omega^2/2\beta^2}.$$

The static characteristic of discriminator in the passband of amplifier is considered linear

$$\psi(\Omega)=S\Omega.$$

Then from expression (1.10) it follows

$$n_x(x_c)=\frac{N_x(x_c)}{K^2_{\phi\kappa}S^2P^2\beta\sqrt{\frac{\pi}{2}}}=$$

$$=\frac{1}{(1+q)^2}\{e^{-x^2/2}+\sqrt{2}\,q\,[(2x_c-x)^2\times$$

$$\times e^{-(x_c-x)^2}+(2x_c+x)^2e^{-(x_c+x)^2}]\},\qquad(1.12)$$

where $x=\Omega/\beta$, $x_c=\Omega_c/\beta$, $P=P_c+P_ш$ — the total power of signal and noise at the entrance of discriminator; $P_c=U_c^2/2$; $P_ш=G_{ш0}\beta/2\sqrt{\pi}$; $G_{ш0}$ — spectral noise density; $q=P_c/P_ш$ — ratio of the power of signal to the power of noise.

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Fig. 1.12 depicts dependences $n_x(x_c)$, calculated by formula (1.12) with the zero detuning of signal $x_c=0$. From the graphs it is possible to find the cut-off frequency, at which the spectral density is in effect constant.

Fig. 1.13 depicts fluctuating characteristics $n_\phi(x_c)$. They can be used for the analysis of the disruption/separation of tracking when

the energy spectrum of output stress/voltage $u_{д}(t)$ of discriminator is uniform in the band of follower. From the graphs it is evident that spectral density $n_{0}(x_{c})$ sharply depends on detuning x_{c} virtually in the entire region of the interesting us signal-to-noise ratios.

Frequently into the circuit of the receiver, which precedes frequency discriminator, for the standardization of power is switched on system ARU or limiter. Performance calculation of discriminator in the presence of inertial system ARU can be carried out through formulas (1.9) and (1.10), if we as the input spectral density use function $N_{1}(\omega)=N(\omega)P_{0}/(P_{c}+P_{ш})$, where P_{0} - power of oscillations, ensured by system ARU [1].

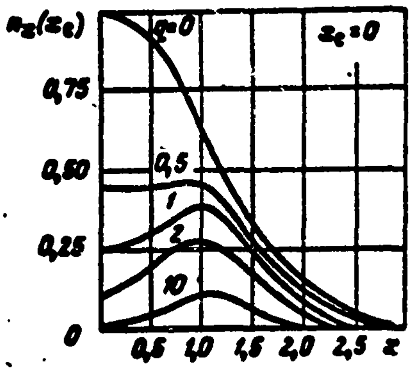

Fig. 1.12.

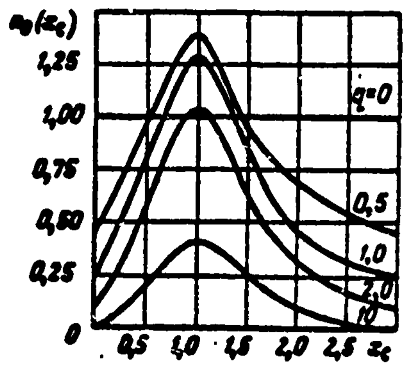

Fig. 1.13.

Fig. 1.12. Spectrum of output potential of frequency discriminator.

Fig. 1.13. Fluctuating characteristics $n_0(x_c)$.

4. Direction finders.

For the isolation/liberation from the signal of information about the angular position of radar target are used the devices/equipment, called direction finders. Recently the widest use received direction finders with the instantaneous comparison of signals. As an example can serve sum-and-difference type direction finder whose simplified functional diagram for one plane of direction finding is depicted in Fig. 1.14.

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The signal, reflected from target, comes simultaneously two antennas with radiation patterns $G_1(\varphi)$ and $G_2(\varphi)$ displaced to the angle 2β. The plumbing, which stands at the input of receiver, forms/shapes total u_Σ and difference u_Δ of stresses/voltages, which together with a stress/voltage of the heterodyne Γ enter mixers SM_1 and SM_2. The stresses/voltages, obtained as a result of conversion, are reinforced by cascades/stages $UPCh_1$ and $UPCh_2$ and enter the phase discriminator FD. For the standardization of received signal in the amplitude in the diagram, depicted in Fig. 1.14, is used instantaneous automatic gain control (MARU). Because of MARU output potential of total channel is kept constant, and the output stress/voltage of difference channel is changed inversely proportional to voltage on the input of total channel. Phase discriminator (FD), implementing the operation of the multiplication of input signals, forms/shapes on the output of direction finder the stress/voltage, proportional to the relation of the stresses/voltages of the difference and total channels

$$u_{вых} \sim \frac{u_\Delta}{u_\Sigma}. \qquad (1.13)$$

The calculation of the discriminatory and fluctuating characteristics of direction finders composes, as a rule, very complex problem, since for this it is necessary to analyze a large quantity of cascades/stages of receiver, including nonlinear. Without

stopping on the details of analysis, let us note that the discriminatory characteristic of the direction finder in question can be obtained from relationship ratio (1.13), if we take into account concrete/specific/actual forms radiation patterns of the antennas of receiver.

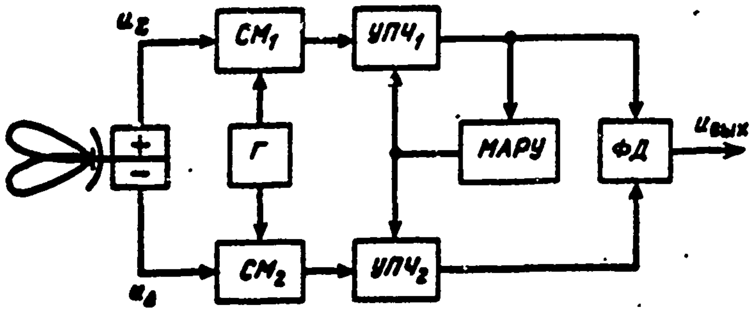

Fig. 1.14. Total-difference type direction finder: SM - mixer; Г - heterodyne; UPCh - IF amplifier; FD - phase discriminator; MARU - diagram of instantaneous automatic gain control.

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As a result the discriminatory characteristic of direction finder will be determined by the expression

$$F(\theta)=U_0\frac{G(\theta-\beta)-G(\theta+\beta)}{G(\theta-\beta)+G(\theta+\beta)}, \qquad (1.14)$$

where 2β - angle between maximums of radiation patterns; θ - current displacement angle between the axis of equisignal sector and the direction of the arrival of signal; U_0 - maximum output potential of phase discriminator, attained at the disagreement/mismatch $\theta=\pm\beta$.

The more detailed calculation of the discriminatory characteristics of the direction finders of different types is, for example, in [91]. In this work let us note only the special feature/peculiarity of discriminatory characteristics (Fig. 1.15),

which consists in the existence of several points of stable and unstable equilibrium. This is explained by the presence of minor lobes in radiation patterns of the antennas of direction finder. The working section of discriminatory characteristic, which has the greatest slope/transconductance, is arranged/located in the vicinity $\theta \sim 0$. It is formed by major lobes of radiation patterns. Side-lobe level of radiation patterns usually is 20-40 dB lower than the level of the main things and therefore in the majority of cases it cannot be taken into consideration. However, sometimes target tracking can be realized by minor lobes, then discriminatory characteristic must be examined in the form, shown in Fig. 1.15.

Together with the direction finder of the type examined in the practice frequently is applied the system with instantaneous amplitude comparison [91], the standardization of signal in which is realized by logarithmic amplifiers. The calculation of the discriminatory and fluctuating characteristics of this direction finder is, for example, in work [106].

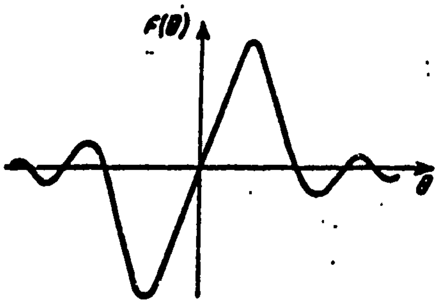

Fig. 1.15. Standard direction-finding characteristic.

1.3. Concept of interruption of tracking.

As was noted in § 1.1, the behavior of servo system can be described stochastic differential equation (1.2), which characterizes change in the time of following error in the regulating circuit. The solution of this equation due to the presence of noise $\xi(t)$ is the random function of time. By analogy with Brownian motion it is possible to say that coordinate x(t) randomly "strays" along the axis $-\infty<x<\infty$.

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However, dependence F(x) can be considered as certain force, which attempts to hold down/retain coordinate x(t) near the point of stable equilibrium of system. If with wandering coordinate x(t) will be beyond the limits of the points γ_1, γ_2 whose coordinates are established/installed previously, then with some stipulations it is

possible to claim that in the system occurred the disruption/separation of tracking. The position of end-points γ_1 and γ_2 in the majority of the cases is determined on the sharp decrease near them of restoring force F(x). This occurs, for example, in the servo auto-selector of impulses/momenta/pulses on distance [44, 62, 75] and in the system of frequency self-alignment [55, 56] whose discriminatory characteristics are depicted respectively in Fig. 1.4 and 1.11. In such systems as a result of the output of coordinate x(t) from the aperture of discriminatory characteristic the ring of automatic control is broken and system becomes unguided.

In some systems of coordinates γ_1, γ_2 correspond to those misalignments x, with which the power of the signal, which passed through the receiver from the ring of automatic control, falls below threshold level. This occurs, for example, in the system of angular target tracking when the receiver of locator additionally is gated on the distance or in the frequency. With the sufficiently large tracking errors on the angle the power of signal in the total channel falls. If the freedom from interference of internal duct/contour is insufficiently high, then with some threshold value of following error on the angle occurs the disruption/separation of range tracking (frequency). This leads to the disappearance of signal at the output of direction finder and to the disruption/separation of tracking by angle [71].

In a number of cases the characteristic of discriminator is periodic function x. If the porosity of characteristic is great, then by disruption/separation of tracking it is possible to understand the output of coordinate x beyond the limits γ_1, γ_2, determined from the decrease of restoring force of F(x) in one period of discriminatory characteristic. This is completely justified, since wandering of coordinate in region $F(x) \approx 0$ occupies usually long time. A similar situation is observed, for example, in the pulse auto-selector with the large porosity of transmitted pulses.

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With low duty factor of the characteristic of discriminator (for example, in the system of phase automatic frequency control) by disruption/separation of tracking frequently is understood [60, 70] the first output of coordinate x(t) for the nearest points of unstable equilibrium on characteristic F(x). This event is occasionally referred to as the disruption/separation of synchronism [54], in contrast to the total loss of tracking the frequency, which occurs only with the repeated migrations/jumps of phase. A question about the disruption/separation of tracking in the systems with the periodic characteristics of discriminators is in more detail examined

in § 3.3.

Strictly speaking, by disruption/separation of tracking should be understood the output of coordinate x(t) beyond the limits of the aperture of discriminatory characteristic to the period, greater than certain permitted for this system. However, the probability of the return of coordinate x(t) for a comparatively short time to the region of tracking is usually small; therefore in such cases with the great probability it is possible to claim that the first output of coordinate x(t) beyond the limits of aperture γ_1, γ_2 is equivalent to the disruption/separation of tracking. The validity of this confirmation increases with the increase of the inertness of regulating circuit. Subsequently, as a rule, by disruption/separation of tracking is understood the first output of coordinate x beyond the limits of the established/installed boundaries γ_1, γ_2.

Let us pause at the fundamental quantitative characteristics of the disruption/separation of tracking. Total characteristic is probability $P(x_0, t)$ of disrupting/separating the tracking for the preset time of observation t. In this case it is assumed that at the initial moment t=0 occurred the mode/conditions of tracking, i.e., $\gamma_1<x_0=x(t=0)<\gamma_2$. Depending on the character of task the initial value of coordinate x_0 can be determined or random.

Sometimes for the characteristic of disruption/separation instead of probability $P(x_*, t)$ is used its derived

$$W_{x_*}(t)=\frac{\partial P(x_*,t)}{\partial t}, \qquad (1.15)$$

being density of distribution of the probability of time to the disruption/separation. However, probability $P(x_*, t)$ can be considered as the integral law of time allocation to the disruption/separation of tracking.

The important parameter, which are determining the quality of follower, is the intensity of fluctuations at the entrance of the system, in which the disruption/separation for the preset time of observation occurs with the probability not more than the given one.

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By intensity of fluctuations, as a rule, is understood the value of spectral density $N_*(x)$ the low-frequency components of noise $\xi(t)$, the led to the output discriminator. The recalculation of this value in the signal-to-noise ratio at the input of discriminator, which is most interesting for the practice, can be carried out with the help of the relationships/ratios, given in § 1.2.

The enumerated characteristics of the disruption/separation of tracking are sufficiently complete; however, their practical

determination is frequently connected with the serious mathematical difficulties. Therefore for the approximate calculations it is expedient to use simpler, although by less general/common/total characteristics. For example, in the strongly inertial systems of tracking with the time of observation, the much larger time of the establishment of transient mode/conditions, the dependence of the probability of disruption/separation on the power of fluctuations carries the character, close to the threshold. In such systems for the approximate computations it is possible to propose [68] that with spectral density, larger certain critical value $N_{кр}$, the disruption/separation of tracking occurs with the probability, close to one, but at the less spectral density - virtually it is not observed. Value $N_{кр}$ in these cases is used for the rough estimate of the quality of the work of servo system.

Approximately the phenomenon of disruption/separation can be characterized also by the first moments/torques of distributing $W_{x_0}(t)$ the time, which passed from the start of system to the disruption/separation of tracking. Important role they here play the mean time $m_1(x_0)$ to the disruption/separation and dispersion $\mathscr{D}(x_0)$ of time to the disruption/separation. The determination of these values in many instances can be carried out by comparatively simple methods. In more detail questions of the determination of the first moments of time to the disruption/separation are examined in Chapter 5.

1.4. Short historical outline.

With the development of vibration theory began to increase the interest in the analysis of the nonlinear dynamic systems, subjected to the action of random interferences. The first works in this direction appeared in the thirties of the current century. Here should be, first of all, noted the basic work of A. A. Andronov, L. S. Pontriagin and A. A. Witt [35], in whom it was for the first time proposed to use for determining the statistical characteristics of dynamic systems an apparatus for the theory of Markov processes. To the success of this approach to a considerable degree contributed the appearing on the eve fundamental work of A. N. Kolmogorov and M. A. Leontovich [25, 26], dedicated to a strict mathematical conclusion/output of equations of Fokker-Planck.

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A considerable effect on further development of the methods of the analysis of the nonlinear systems, subjected to the action of random disturbances, had the works of S. M. Rytov, I. L. Bershteyn, P. I. Kuznetsov, R. L. Stratonovich, V. I. Tikhonov, P. S. Landy. Among them should be isolated articles [23, 28, 36, 37, 50]. Fruitful

proved to be also the ideas of the work of Kramers [27] and Chandrasekhar [20], the dedicated to the study of diffusion Brownian particles in nonuniform field of force.

For the first time the task of the analysis of the disruption/separation of tracking in the radio engineering regulating circuits formulated, apparently, A. M. Vasil'yev [44]. After noting the analogy between the behavior of Brownian particles and random change of the following error in the regulating circuit, A. M. Vasil'yev succeeded in using for the analysis of the disruption/separation of tracking the apparatus of the diffusion equations of Fokker-Planck. To a number of first works according to the analysis of the disruption/separation of tracking belongs also the work of I. A. Bol'shakov [46], in whom with the help of Peetz-Galerkin method is found the approximation for the probability of disruption/separation in the nonlinear first-order system.

Considerable attention to the problem of the disruption/separation of tracking was given in the international congresses for the automatic control (IFAK), where among others were represented reports [32, 48, 49]. Thematics of the majority of reports, as a rule, did not exceed the scope of the examination of first-order systems.

The significant contribution to the research of the equations of Fokker-Planck introduced the monograph of R. L. Stratonovich [14] left in 1961, who played noticeable role and in the development of applied questions of the theory of Markov processes.

Beginning with 1959-1961 the analysis of the disruption/separation of tracking in the radio engineering systems it is developed especially ra; dly, store/add up the basic schools, which work in this direction. One of them, headed by V. I. Tikhonov, successfully works in the region of the analysis of the statistical characteristics of the systems of phase automatic frequency control [45, 52-54, 64, 69, 80, etc.]. The analysis of stability of the pulse servo system in the conditions for noise effect is carried out in the work of I. N. Amiantov and V. I. Tikhonov [21]. In the work of V. I. Tikhonov [47] is for the first time examined the system FAPCh, described stochastic differential second order equation. The detailed survey/coverage of works up to 1964, dedicated to the analysis of the statistical characteristics of different systems FAPCh, is given in article [60].

The work of another school [55, 62, 63, 67, 71, 84, etc.], created by V. L. Lebedev, are dedicated to the analysis of disruption/separation in different servo systems of the first and second order. In particular, in works [55, 62, 67] considerable

attention is given to the research of the disruption/separation of tracking by conducting the analogy with the Brownian particles, the surmounting particles, which surmount potential Barber. In the article of S. V. Pervachev [62] is for the first time correctly posed the problem about the disruption/separation of tracking in the system of the second order with the proportional-integrating filter, is comprised for this case the equation of Fokker-Planck and for a series/row of special cases is obtained its approximate solution.

To the analysis of the statistical characteristics of systems FAPCh is dedicated the series/row of the works of V. V. Shakhgil'dyan [70, 77, etc.].

To the determination of the approximate stall conditions of tracking in the complicated nonlinear systems is given much attention in the work of collective under G. G. Sigalov's management/manual [68, 85, etc.].

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Taking into account that the dependence of the probability of disrupting/separating the tracking on the signal-to-noise ratio in a number of cases carries a sharply pronounced threshold character, the authors of these works, using theory of statistical linearization and

averaged differential equations, define critical stall conditions as loss of stability in the system. To analogous questions are dedicated also works [69, 88]. Unfortunately, the estimations, found with such methods, lose their statistical properties and do not depend on the time of noise effect in the system.

In the works of V. M. Artem'yev [89, etc.] to the analysis of the disruption/separation of tracking extends one of the modifications of the method of successive approximations - method of the averaging of functional corrections which is used for approximate solution of the equation of Fokker-Planck. However, unwieldiness of method substantially impedes its use in the practice.

Among the foreign research in the field of the analysis of the disruption/separation of tracking it is possible to note works [51, 57, 58, 66, 76], dedicated in essence to the study of different systems of phase automatic frequency control.

On the formulation of the problem to the analysis of disruption/separation is close the task of the definition of the characteristics of the ejections of noise for certain level. To detailed research of these questions is dedicated, in particular, monograph [17].

Development of analog and digital computational technology made it possible to work out the series/row of the methods of determining the probability of disruption/separation with the help of the simulation of servo systems in the computers. In works [32, 82, 90] is demonstrated the possibility of determining the statistical characteristics of servo systems by the method of solution in the analog and digital computers of the corresponding equations of Fokker-Planck and Pontriagin.

To experimental resear f the disruption/separation of tracking in different radic ,ineering regulating circuits are dedicated works [52, 53, 56, 58, 61, 76].

Certain representation about the history of the development of the methods of the analysis of the disruption/separation of tracking can be obtained from the section Б of the bibliography, placed at the end of the book. Bibliography in this section is comprised in the chronological order.

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Chapter 2.

BASIC INFORMATION FROM THE THEORY OF MARKOV PROCESSES.

The theory of Markov processes in spite of a comparative youth is the very developed region of mathematics and plays large role in the research of nonlinear regulating circuits. Without having the capability to state it in detail, let us pause at some most important positions, necessary for the analysis of the disruption/separation of tracking.

2.1. Basic concepts. Terminology.

Concept of aftereffect. For determining the Markov process high value has a concept of aftereffect. Random process x(t) is characterized by m-dimensional probability density $w(x_1, x_2, \ldots, x_m; t_1, t_2, \ldots, t_m)$, where x_i— value of process of x(t) at the moment of time t_i.

Let us introduce into the examination conditional density $w(x_m, t_m | x_1, t_1; x_2, t_2; \ldots x_{m-1}, t_{m-1})$, which characterizes the distribution of process of x(t) at the moment of time t_m, if at the previous moments/torques it took values $x_1, x_2, \ldots, x_{m-1}$. This makes it possible to register

$$w(x_1, x_2, \ldots, x_m) = w(x_1, x_2, \ldots, x_{m-1}) \times \times w(x_m | x_1, \ldots, x_{m-1}). \quad (2.1)$$

Here and sometimes subsequently for the reduction of recording temporary/time arguments in the distribution functions lower.

Special interest they present two special cases:

$$1) \quad w(x_m | x_1, \ldots, x_{m-1}) = w(x_m). \quad (2.2)$$

The given relationship/ratio characterizes the mutual independence of separate ordinates x_i of process x(t).

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In this case multidimensional probability density falls into the product of the one-dimensional ones

$$w(x_1, x_2, \ldots, x_m) = w(x_1) w(x_2) \ldots w(x_m).$$

The process, which possesses property (2.2), is called white noise.

$$2) \quad w(x_m | x_1, x_2, \ldots, x_{m-1}) = w(x_m | x_{m-1}). \quad (2.3)$$

Relationship/ratio (2.3) characterizes the simplest form of communication between the separate ordinates of process x(t), i.e., the value of ordinate at the m moment of time depends only on the value of ordinate at the previous moment/torque. This random process is conventionally designated as process without the aftereffect or by Markovian (on the name of A. A. Markov, who for the first time studied the discrete/digital version of this process).

Markov process is completely characterized by two-dimensional probability density, or it is more precise, by one-dimensional density and with the probability density of transition. Actually/really, on the basis of (2.1) taking into account (2.3) we obtain

$$\begin{aligned} w(x_1, \ldots, x_m) &= w(x_1, \ldots, x_{m-1})\, w(x_m \mid x_{m-1}) = \\ &= w(x_1, \ldots, x_{m-2})\, w(x_{m-1} \mid x_{m-2})\, w(x_m \mid x_{m-1}) = \\ &= w(x_1)\, w(x_2 \mid x_1) \ldots w(x_m \mid x_{m-1}). \end{aligned} \quad (2.4)$$

Function $w(x_j|x_i) = w(x_j, t_j; x_i, t_i)$ is a probability density of the transition of process of x(t) from state x_i, of occurred at the moment time t_i, into state x_j up to the moment/torque of time t_j.

Markov process is conveniently examined in the phase space Ω whose dimensionality is determined by number n of mutually independent coordinates $x_1, x_2, \ldots, x_n$. If n=1, the Markov process x(t) is called one-dimensional, if n>1 - multidimensional. In the latter case the state of process x(t) at the moment of time t_k is

characterized by vector $x_k=\{x_{1k}, x_{2k}, \ldots, x_{nk}\}$, determined in the phase space Ω. Then the probability density of transition is written/recorded as $w(x_j, t_j; x_i, t_i)$.

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It is possible to show [5] that process x(t) is n-dimensional Markovian, if its components x_i satisfy the system stochastic differential equations

$$\frac{dx_i}{dt}=a_i(x_1, x_2, \ldots, x_n, t)+\sum_{j=1}^{n} b_{ij}(x_1, x_2, \ldots, x_n, t)\xi_j^0(t), \quad (2.5)$$
$$i=1, 2, \ldots, n,$$

where a_i, b_{ij}— determined functions, in the general case nonlinear, $\xi_j^0(t)$—independent white noises with the single spectral densities. In expression (2.5) the spectral density of real random process always can be reduced to the single by the corresponding change in the coefficients of intensities b_{ij}. However, this requirement is not fundamental and is introduced only for convenience in further recording.

A question about how to determine multidimensional Markov process so that the following error would be one of its component, is examined in the following paragraph.

Are at present known sufficient conditions with executing of which there is continuous and unique solution of the system stochastic equations (2.5) [5]. These conditions limit an increase in coefficients $a_i(x, t)$ and $b_{ij}(x, t)$. For the one-dimensional Markov process, for example, must exist such $M<\infty$, so that so on of all $x\in\Omega$ and $y\in\Omega$ would be satisfied the conditions

$$|a(x, t)-a(y, t)|+|b(x, t)-b(y, t)|\leqslant M|x-y|,$$
$$a^2(x, t)+b^2(x, t)\leqslant M^2(1+x^2).$$

With the disturbance of these conditions for existence of the unique and continuous solution stochastic equation must be proved additionally. In the tasks about the disruption/separation of tracking stochastic equation is assigned in the limited interval of values x; therefore the formulated conditions, as a rule, are satisfied for the real forms of discriminatory and fluctuating characteristics.

Let us pause at some important properties of Markov processes.

Stability. Markov process is called stationary (uniform), if the probability density of transition $w(x_j, t_j; x_i, t_i)$ depends only about difference $\tau=t_j-t_i$ and it does not depend on the position of the initial moment of reading t_i.

Relationship/ratio of Chapman-Smoluxovsky. Large role in the

theory of Markov processes plays the relationship/ratio of Chapman-Smoluxovsky.

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If we introduce three moments/torques of time t_0, t' and t, such, that $t_0 < t' < t$, that this relationship/ratio takes form [18]

$$w(x, t; x_0, t_0) = \int_{-\infty}^{\infty} w(x', t'; x_0, t_0)\, w(x, t; x', t')\, dx'. \quad (2.6)$$

It makes it possible to determine the probability density of the transition of process of x(t) from state x_0 into state x, if they are known to the probability density of transitions from x_0 to the intermediate state x' and from x' in x.

2.2. Description of control systems with the help of the Markov processes.

In the servo radio engineering systems the random process x(t) being investigated (for example, the current error of automatic tracking) is assigned stochastic differential equation of form (1.2). In order to have the capability to study the behavior of process x(t) by the methods of the theory of Markov processes, it is necessary to, first of all, express $x(t)$ through the components of the corresponding Markov process x(t), in the general case of

multidimensional. In other words, it is necessary to select such coordinates of n-dimensional phase space Ω so that x(t) in this space would prove to be process without the aftereffect. For this is necessary satisfaction of the following conditions.

First, all random disturbances, entering initial equation (1.2), must take the form of white noises. In the second place, should be so selected the coordinates of phase space, in which is determined vector x(t), so that n-dimensional equation (1.2) it would be possible to register in the form of system (2.5) stochastic first-order equations. The first condition usually is satisfied, since the servo systems in the majority of the practical cases have the narrow passband in limits of which the interference spectrum it is possible to consider uniform. In such a case, when the spectrum of disturbance/perturbation η(t) is substantially nonuniform in the passband of system, the introduction of the further forming filter (Fig. 2.1) with the operational gear ratio/transmission factor $K_\xi(p)$ makes it possible to reduce the perturbing action to the white noise ξ(t). As a result initial stochastic equation (1.2) will take the form

$$x(t)=\lambda(t)-K(p)[F(x)+K_\xi(p)\xi(t)]. \qquad (2.7)$$

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The introduction of the forming filter leads to an increase in

the order stochastic differential equation and, therefore, to the complication of the process of determining the probability of disruption/separation.

Satisfaction of the second condition, as a rule, causes great difficulties and requires further limitations to the form of differential equation (2.7). In particular, for this it is necessary that the gear ratios/transmission factors K(p) and $K_\xi(p)$ would be the rational-fractional functions of operator p.

Known several methods of the determination of phase coordinates $x_1, x_2, \ldots, x_n$, which make it possible to introduce the Markov process of x(t), connected with random process of x(t) [10, 24, 39, 42, 57, 62]. Let us consider some of them.

First method. Let $K(p)=L_0(p)/M_0(p)$ и $K_\xi(p)=L_1(p)/M_1(p)$m where L_0, L_1, M_0 and M_1 - polynomials of the degrees cf operator p. Let us register equation (2.7) in the following form:

$$M(p)x(t)=M(p)\lambda(t)-L(p)F(x)-Q(p)\xi(t), \quad (2.8)$$

where $M(p)=M_0(p)M_1(p)=\mu_n p^n+\mu_{n-1}p^{n-1}+\ldots+\mu_0$ — polynomial of the n degree relative to $L(p)=L_0(p)M_1(p)=l_m p^m+\ldots+l_0$ and $Q(p)=L_0(p)L_1(p)=q_r p^r+\ldots q_0$ — polynomials are not older than the (n-1)th degree. Without the limitation of generality let us assume $\mu_n=1$.

Let us isolate separately the case, when Q(p) is the polynomial of zero degree ($Q(p)=q_0$).

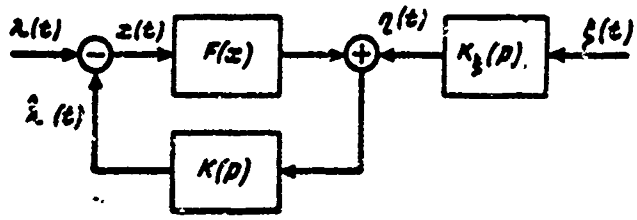

Fig. 2.1. Bringing random disturbance to the white noise.

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In this case the following error x(t) is one of the components of the n-dimensional Markov process x(t) in the space with traditional phase coordinates $x_1, x_2, \ldots, x_n$:

$$\left.\begin{aligned} &x_1 = x(t),\\ &\frac{dx_1}{dt} = x_2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\frac{dx_{n-1}}{dt} = x_n,\\ &\frac{dx_n}{dt} = -\mu_{n-1}x_n - \ldots - \mu_0 x_1 +\\ &+ M(p)\lambda(t) - L(p)F(x) - q_0\xi(t). \end{aligned}\right\} \quad (2.9)$$

with $r > 1$ the introduction of multidimensional Markov process is complicated. Let us consider the preliminarily special case when dependence F(x) is linear, i.e., F(x)=Sx (linear discriminator). According to [10] let us introduce the first n-1 coordinates of phase space Ω that so that

$$\left.\begin{aligned}&x_1=x(t),\\&\frac{dx_1}{dt}=x_2+C_1\xi(t),\\&\frac{dx_2}{dt}=x_3+C_2\xi(t),\\&\dots\dots\dots\dots\\&\frac{dx_{n-1}}{dt}=x_n+C_{n-1}\xi(t),\end{aligned}\right\}\qquad(2.10)$$

where $C_1, C_2, \ldots, C_{n-1}$ — some, unknown thus far coefficients.

From (2.10) it follows that

$$\begin{aligned}&\frac{d^kx}{dt^k}=x_{k+1}+\sum_{i=0}^{k-1}C_{k-i}\frac{d^i\xi(t)}{dt^i}\quad\text{при}\quad 1\leqslant k\leqslant n-1,\\&\frac{d^nx}{dt^n}=\frac{dx_n}{dt}+\sum_{i=1}^{n-1}C_{n-i}\frac{d^i\xi(t)}{dt^i}.\end{aligned}\qquad(2.11)$$

For the linear function F(x)=Sx is fulfilled the relationship/ratio

$$L(p)F(x)=SL(p)x.\qquad(2.12)$$

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Taking into account (2.11) and (2.12) equation (2.8) takes the form

$$\begin{aligned}\mu_n\frac{dx_n}{dt}=&-\sum_{k=0}^{n-1}\mu_kx_{k+1}-\sum_{i=0}^{n-2}\frac{d^i\xi(t)}{dt^i}\sum_{k=1}^{n-i-1}\mu_{k+i}C_k-\\&-\mu_n\sum_{i=1}^{n-1}C_{n-i}\frac{d^i\xi(t)}{dt^i}+M(p)\lambda(t)-S\sum_{k=0}^{m}l_kx_{k+1}-\\&-S\sum_{i=0}^{m-1}\frac{d^i\xi(t)}{dt^i}\sum_{k=1}^{m-i}l_{k+i}C_k-\sum_{i=0}^{r}q_i\frac{d^i\xi(t)}{dt^i}.\end{aligned}\qquad(2.13)$$

Let us select coefficients C_i so that the factors, which stand in equation (2.13) with the derivatives of white noise, would become zero. For this is necessary satisfaction of the following conditions:

$$\sum_{k=1}^{n-i-1} \mu_{k+i} C_k + \mu_n C_{n-i} + S \sum_{k=1}^{m-i} l_{k+i} C_k + q_i = 0, \quad (2.14)$$

$$i = 1, 2, \ldots, n-1.$$

Sequentially solving equations (2.14), let us determine unknown coefficients $C_1, C_2, \ldots, C_{n-1}$. with the execution of equalities (2.14) equation (2.13) takes the form

$$\mu_n \frac{dx_n}{dt} = M(p)\lambda(t) - \sum_{k=0}^{n-1} \mu_k x_{k+1} - S \sum_{k=0}^{m} l_k x_{k+1} -$$

$$- \left[\sum_{k=1}^{n-1} \mu_k C_k + S \sum_{k=1}^{m} l_k C_k + q_0 \right] \xi(t). \quad (2.15)$$

Latter/last equation and equations (2.10) form the unknown system

stochastic differential equations relatively component $x_1, x_2, \ldots, x_n$ of the n-dimensional Markov process x(t).

With the nonlinear characteristic of discriminator F(x) of equation (2.8) also it is possible to reduce to the system stochastic first-order equations, not containing derivatives of white noise.

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S. V. Pervachev proposed the following method of the introduction of the first n-1 the component of the Markov process

$$\left.\begin{aligned}
&x_1 = x(t),\\
&\frac{dx_1}{dt} = x_2,\\
&\cdots\cdots\cdots\cdots\\
&\frac{dx_{n-s-1}}{dt} = x_{n-s},\\
&\frac{dx_{n-s}}{dt} = x_{n-s+1} + C_1\xi(t),\\
&\frac{dx_{n-s+1}}{dt} = x_{n-s+2} + C_2(x_1)\xi(t),\\
&\frac{dx_{n-s+2}}{dt} = x_{n-s+3} + C_3(x_1, x_2)\xi(t),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot;\\
&\frac{dx_n}{dt} = x_n' + C_s(x_1, x_2, \ldots, x_{s-1})\xi(t),
\end{aligned}\right\} \quad (2.16)$$

where s= max. (m, r) - the greatest exponent of polynomials L(p) and Q(p).

In contrast to system (2.10) here factors C_i are the functions of variable/alternating x_j. The general/common/total methodology of the determination of functions $C_i(x)$ is analogous to the methodology of that [illegible]sed above of the determination of coefficients C_i in equalities (2.10).

Let us consider the example very widespread in practice.

Example. Let the feedback loop of the ring of automatic control consist of integrator and proportional-integrating filter, so that

$$K(p)=\frac{K}{p}\frac{1+pT_1}{1+pT},\ \alpha=\frac{T_1}{T}. \qquad (2.17)$$

The perturbing action let us represent in the form $\xi(t)=\sqrt{N_0(x)}\xi^0(t)$, where $\xi^0(t)$ — white noise with the single spectral density. Then differential equation (1.2) will take the form

$$T\frac{d^2x}{dt^2}+\left(1+K\alpha T\frac{dF(x)}{dx}\right)\frac{dx}{dt}+KF(x)=T\frac{d^2\lambda}{dt^2}+$$
$$+\frac{d\lambda}{dt}-K\sqrt{N_0(x)}\xi^0(t)-K\alpha T\frac{d[\sqrt{N_0(x)}\xi^0(t)]}{dt}. \qquad (2.18)$$

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According to (2.16) let us introduce new variables x_1, x_2:

$$x_1=x(t),$$
$$\frac{dx_1}{dt}=x_2+C\sqrt{N_0(x_1)}\xi^0(t). \qquad (2.19)$$

After substituting (2.19) in initial equation (2.18), we will obtain the second equation

$$\frac{dx_2}{dt}=-\frac{1}{T}\left(1+KnT\frac{dF(x_1)}{dx_1}\right)x_2-\frac{K}{T}F(x_1)+$$
$$+\frac{d^2\lambda}{dt^2}+\frac{1}{T}\frac{d\lambda}{dt}-\frac{\sqrt{N_0(x_1)}}{T}\left[K+C\left(1+KnT\frac{dF(x_1)}{dx_1}\right)\right]\times$$
$$\times\xi^0(t)-(Kn+C)\frac{d\left[\sqrt{N_0(x_1)}\,\xi^0(t)\right]}{dt}. \qquad (2.20)$$

Factor C is determined from the condition of equality to zero coefficients with the derivative of white the bag:

$$C=-Kn.$$

As a result the system stochastic equations relatively component x_1, x_2 of two-dimensional Markov process takes form [62]:

$$\left.\begin{aligned}&\frac{dx_1}{dt}=x_2-Kn\sqrt{N_0(x_1)}\,\xi^0(t),\\&\frac{dx_2}{dt}=-\frac{1}{T}\left(1+KnT\frac{dF(x_1)}{dx_1}\right)x_2-\frac{KF(x_1)}{T}+\\&+\frac{d^2\lambda}{dt^2}+\frac{1}{T}\frac{d\lambda}{dt}-\frac{K}{T}\sqrt{N_0(x_1)}\left(1-n-Kn^2T\times\right.\\&\left.\times\frac{dF(x_1)}{dx_1}\right)\xi^0(t).\end{aligned}\right\} \qquad (2.21)$$

As it follows from (2.19), in the method of replacing the variable/alternating examined the studied random process x(t) corresponds to one component $x_1(t)$ the introduced two-dimensional Markov process x(t).

The second method of the composition of the system stochastic equations was examined by J. Dub [24] for the linear systems and was spread by E. Viterbi [57] to the nonlinear followers. Let us consider one of the modifications of this method where in contrast to [57] is considered the action of the determined disturbance/perturbation $\lambda(t)$.

We will be bounded to the analysis of the situation when the random disturbance, converted to the output of the discriminator (see Fig. 2.1), it is possible to represent in the form of the white noise $(K_\xi(p)=1)$[1].

FOOTNOTE [1]. When $K_\xi(p)\neq 1$ to the system of equations, which characterize the ring of tracking, should be supplemented the equations, which describe the forming filter. ENDFOOTNOTE.

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Under this condition equation (2.8) takes the form

$$M_0(p)[x(t)-\lambda(t)]=-L_0(p)[F(x)+\xi(t)].$$

Let us introduce the new variable/alternating $x_1(t)$ so, in order to

$$x(t)-\lambda(t)=\frac{1}{b_0}L_0(p)x_1(t).$$

Joining two latter/last relationships/ratios, we will obtain the

equation

$$M_0(p)x_1(t) = -l_0F(x) - l_0\xi(t),$$

to which corresponds the following system stochastic equations:

$$\left.\begin{aligned}&\frac{dx_1}{dt}=x_2,\\&\dots\dots\dots\dots\dots\dots\\&\frac{dx_{n-1}}{dt}=x_n,\\&\frac{dx_n}{dt}=-\mu_0x_1-\dots-\mu_{n-1}x_n-l_0F(x)-l_0\xi(t).\end{aligned}\right\}\quad(2.22)$$

In these expressions it is taken into consideration, that $M_0(p)=\sum_{i=0}^{n}\mu_ip^i, L_0(p)=\sum_{i=0}^{m}l_ip^i,\ \mu_n=1.$ As a result of the done replacement of variable/alternating the random process x(t) being investigated it is possible to represent by linear combination of components $x_i(t)$ and dynamic disturbance/perturbation λ(t):

$$x(t)=\frac{1}{l_0}\sum_{0}^{m}l_ix_{i+1}(t)+\lambda(t).$$

It is here assumed that are known initial conditions $x_1(0), \dots, x_n(0)$, which occurred upon the inclusion of system into moment/torque t=0.

The advantage of the method of the introduction of phase coordinates examined in comparison with the first is the absence in equations (2.22) of derivatives of characteristic f(x). Furthermore, white noise ξ(t) enters only into one equation of system (2.22).

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As shown in S 2.3, these facts conduct to considerable simplification in the corresponding equation of Fokker-Planck.

However, to in practice use the method of replacing the coordinates examined is inconvenient, since disturbance/perturbation λ(t) enters into the dependence, which connects components x_i with process of x(t). With λ(t) $\neq$ const the domain of definition of boundary-value problem for the equation of Fokker-Planck, comprised for coordinates x_i, is changed in the time.

Of the deficiency/lack indicated it is possible to get rid in the particular, but sufficiently spread case when dynamic disturbance/perturbation λ(t) is approximated by the polynomial

$$\lambda(t)=\lambda_0+\lambda_1 t+\frac{\lambda_2}{2!}t^2+\ldots+\frac{\lambda_s}{s!}t^s,$$

old degree s of which does not exceed the order of astaticism of regulating circuit. Let us recall that for the system, which possesses astaticism of the s order, coefficients

$\mu_0=\mu_1=\ldots=\mu_{s-1}=0$. In this case the processes, which take place in the regulating circuit, will not be changed, if dynamic disturbance/perturbation λ(t) is replaced with constant stress U, applied to the output of discriminator additively with the random stress/voltage ξ(t), and we by correspondingly change initial

conditions in the system. The equation of control system in the new coordinates takes the form

$$-z_1(t)=\frac{l_0}{M_0(p)}[F(x)+U+\xi(t)].$$

Hence we will obtain the system stochastic equations:

$$\left.\begin{aligned}&\frac{dz_1}{dt}=z_2,\\&\dots\dots\dots\dots\dots\dots\\&\frac{dz_{n-1}}{dt}=z_n,\\&\frac{dz_n}{dt}=-\mu_0 z_1-\dots-\mu_{n-1}z_n-l_0F(x)-\\&\qquad-l_0U-l_0\xi(t),\end{aligned}\right\}\qquad(2.23)$$

where the following error is expressed only through coordinates z_i:

$$x(t)=\frac{L_0(p)}{l_0}z_1(t)=\frac{1}{l_0}\sum_{i=0}^{m}l_iz_{i+1}(t).$$

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Constant stress $U=-\mu_0\lambda_0/l_0$ and changed initial conditions $z_i(0)$ in the system form at the output of ripple filter signal - $\lambda(t)$, which is equivalent to action at the entrance of the system of signal $\lambda(t)$. Let us designate increases in the initial conditions through ε_i:

$$z_i(0)=x_i(0)+\varepsilon_i,\ i=1,\ 2,\ \dots,\ n.\qquad(2.24)$$

In order to form at the entrance of discriminator polynomial $\lambda(t)$ the s order, it is necessary in accordance with (2.24) to change

initial conditions on s integrators to values $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s$ respectively and feed to the entrance of chain/network from s integrators constant voltage ε_{s+1}. Initial conditions $x_{s+2}, \ldots, x_n$ are not changed, so that $\varepsilon_{s+2}=\varepsilon_{s+3}=\ldots=\varepsilon_n=0$. So that the increases $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{s+1}$ would form signal $\lambda(\tau)$ must be implemented the equality

$$\left(1+\frac{l_1}{l_0}p+\ldots+\frac{l_m}{l_0}p^m\right)\left(\varepsilon_1+\varepsilon_2 t+\ldots+\frac{\varepsilon_{s+1}}{s!}t^s\right)=$$
$$=\lambda_0+\lambda_1 t+\ldots+\frac{\lambda_s}{s!}t^s.$$

Equalizing coefficients with the identical degrees of t, we will obtain system of equations

$$\lambda_k=\varepsilon_{k+1}+\frac{l_1}{l_0}\varepsilon_{k+2}+\frac{l_2}{l_0}\varepsilon_{k+3}+\ldots+\frac{l_{s-k}}{l_0}\varepsilon_{s+1},$$
$$k=0, 1, 2, \ldots, s,$$

which consistently is permitted:

$$\varepsilon_{s+1}=\lambda_s,$$
$$\varepsilon_s=\lambda_{s-1}-\frac{l_1}{l_0}\lambda_s,$$
$$\varepsilon_{s-1}=\lambda_{s-2}-\frac{l_1}{l_0}\lambda_{s-1}+\frac{l_1^2-l_2}{l_0}\lambda_s,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus, values ε_k can be calculated according to the recurrent equations

$$\varepsilon_k=\lambda_{k-1}-\frac{l_1}{l_0}\varepsilon_{k+1}-\frac{l_2}{l_0}\varepsilon_{k+2}-\ldots-\frac{l_{s-k}}{l_0}\varepsilon_{s+1}, \quad (2.25)$$
$$k=s+1, s, \ldots, 1.$$

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Example. For the servo system with the integrator and the proportional-integrating filter, let us make the replacement of the variable/alternating z_1, z_2 according to formulas (2.23) under the effect of $\lambda(t)=\lambda_0+\lambda_1 t$. This leads to the following systems of equations

$$\left.\begin{aligned}\frac{dz_1}{dt}&=z_2,\\ \frac{dz_2}{dt}&=-\frac{1}{T}z_2-\frac{K}{T}F(x)+\frac{1}{T}\lambda_1-\frac{K}{T}\sqrt{N_0(x)}\xi'(t),\end{aligned}\right\}\quad(2.26)$$

where $$x(t)=z_1(t)+T_1 z_2(t).$$

In accordance with (2.24)-(2.25) the initial conditions are connected with the following relationships:

$$z_1(0)=x_1(0)+\lambda_0-T_1\lambda_1,$$
$$z_2(0)=x_2(0)+\lambda_1.$$

From this we determine the initial value of the tracking error at point in time after closing the tracking ring $x(0)=\lambda_0+x_1(0)+T_1x_2(0)$.

are given the basic facts from the theory of the equations of Fokker-Planck.

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Determination. Let n-dimensional Markov process $\mathbf{x}(t)=\{x_1(t), x_2(t), \ldots, x_n(t)\}$ be described by the system stochastic equations

$$\frac{dx_i}{dt}=a_i(\mathbf{x}, t)+\sum_{j=1}^{n} b_{ij}(\mathbf{x}, t)\xi_j^0(t), \quad i=1, 2, \ldots, n. \qquad (2.5)$$

Then the probability density w(x, t) of continuous Markov process satisfies the equation of Fokker-Planck:

$$\frac{\partial w}{\partial t}+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[A_i(\mathbf{x}, t)\, w]=\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}B_{i,j}(\mathbf{x}, t)\, w]. \quad (2.27)$$

Equation (2.27) is linear equation in the partial derivatives with the variable coefficients. The type of equation is determined by the matrix/die of coefficients at the second derivatives B. If matrix B is nondegenerate, the equation of Fokker-Planck relates to the parabolic type. If matrix/die is degenerated in certain point or set of points, then equation relates to ultra-parabolic (elliptic-parabolic) type [34].

The coefficients of equation (2.27)

$$A_i(x, t)=\lim_{\tau\to 0}\frac{\overline{[x_i(t+\tau)-x_i(t)]}}{\tau} \qquad (2.28)$$

characterize local average/mean rate of change in coordinate x_i, and

$$B_{ij}(\mathbf{x}, t) = \lim_{\tau \to 0} \frac{\overline{[x_i(t+\tau) - x_i(t)][x_j(t+\tau) - x_j(t)]}}{\tau} \quad (2.29)$$

- correlation of component x_i and x_j.

In the tradition, which arose during the study of the behavior of Brownian particles, coefficients A_i and B_{ij} are called respectively the coefficients of removal/drift and diffusion. They are determined from the system stochastic equations (2.5). The formal solutions of this system are expressed by the following integral equalities:

$$x_i(t) = x_i(0) + \int_0^t a_i(\mathbf{x}(\tau), \tau) d\tau + \\ + \sum_{j=1}^{n} \int_0^t b_{ij}(\mathbf{x}(\tau), \tau) \xi_j^0(\tau) d\tau. \quad (2.30)$$

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Not examining in detail, let us note that there are two methods of calculating stochastic integrals [5, 15] entering in (2.30). If we use for the calculation the method, proposed by K. Ito [5, 40], then we will obtain that the coefficients stochastic equation (2.5) and equation of Fokker-Planck (2.27) are connected with the relationships/ratios

$$A_i(x, t) = a_i(x, t), \qquad (2.31)$$

$$B_{ij}(x, t) = \frac{1}{2}\sum_{k=1}^{n} b_{ik}(x, t)\, b_{jk}(x, t). \qquad (2.32)$$

In this case it is assumed that spectral density $N_*(x)$ of random process is connected with the correlation function $r(\tau)$ with Fourier transform (1.1).

But if we use determination stochastic integral in the symmetrized form, proposed by R. L. Stratonovich [15], then we obtain another form of the recording of the coefficient of the removal/drift:

$$A_i(x, t) = a_i(x, t) + \frac{1}{4}\sum_{l, k=1}^{n} \frac{\partial b_{ik}(x, t)}{\partial x_l}\, b_{lk}(x, t). \qquad (2.33)$$

Difference in the forms of the recording of the coefficients of removal/drift in the methods of K. Ito and R. L. Stratonovich let us clarify based on the example of the one-dimensional Markov process x(t), assigned stochastic equation

$$\frac{dx}{dt} = a(x) + b(x)\xi^\circ(t).$$

This equation is conveniently represented in the form

$$dx = a(x)dt + b(x)d\zeta,$$

where $d\zeta = \xi^\circ(t)\,dt$ - differential of single Wiener process. Passing to the finite increments, let us register

$$\Delta x = a(x)\Delta t + \frac{da(x)}{dx}\Delta x\Delta t + \ldots + b(x)\Delta\zeta + \frac{db(x)}{dx}\Delta x\Delta\zeta + \ldots$$

Regarding (2.28) the coefficient of removal/drift is equal to

$$A(x) = \lim_{\Delta t\to 0}\frac{\overline{\Delta x}}{\Delta t} = a(x) + \lim_{\Delta t\to 0}\frac{db(x)}{dx}\frac{\overline{\Delta x\Delta\zeta}}{\Delta t};$$

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During the treatment stochastic integral according to K. Ito, the increases Δx and $\Delta\zeta$ are independent random quantities, also, in this case

$$A(x) = a(x).$$

Physically this result is feasible, if one assumes that in the feedback loop of regulating circuit is inherent the delay to the period, greater than the time of the correlation of random disturbance $\xi'(t)$ [1, 30].

During the treatment according to R. L. Stratonovich there is no such delay, therefore as shown, for example, in [30]

$$\lim_{\Delta t\to 0}\frac{\overline{\Delta x\Delta\zeta}}{\Delta t} = \frac{1}{4}b(x),$$

as a result of what the coefficient of removal/drift is equal to

$$A(x) = a(x) + \frac{1}{4}b(x)\,b'(x).$$

At present it is not yet produced a unique solution about the

advisability of the selection of that or another form of the recording of the coefficient of removal/drift in the equations of Fokker-Planck, that corresponds to the physical model in question. Apparently, the use/application of a concrete/specific/actual form of recording must be produced on the basis of the comparison of the time of the correlation of real broadband random process $\xi^{\bullet}(t)$ and signal lag in the ring of automatic control which usually accompanies the passage of signal in the radio engineering circuits.

Subsequently is used, in essence, the form of the recording of the coefficients of removal/drift, proposed by R. L. Stratonovich. In the particular case of $b(x, t)=b(t)$ both forms of the recording of the coefficients of removal/drift coincide.

Flow of probability. In order to determine this concept, let us lead the analogy between the behavior of the trajectories of process $x(t)$ in the phase space Ω and the Brownian motion.

Let us assume that $w(x, t)$ - the concentration of the diffusing Brownian particles. Let us take any volume V with surface of S and it is computed a quantity of particles, passing through the element/cell of surface ΔS for the time Δt.

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Since $A = \|A_i(x, t)\|$ — vector of the average speeds, the quantity of particles, passing through ΔS for the time Δt due to the convection, is proportional to scalar product Aw to the external normal n to the element/cell of surface ΔS:

$$(Aw, n)\Delta S\Delta t. \qquad (2.34)$$

Let us assume for simplicity that B(x)=B and let us consider the quantity of particles, passing through ΔS due to the diffusion. It is proportional normal derivative concentration of substance ∂w/∂n:

$$-\frac{1}{2}B\frac{\partial w}{\partial n}\Delta S\Delta t. \qquad (2.35)$$

Minus sign means that the diffusion occurs in the direction from the larger concentration of substance to smaller. Let us consider a change of the number of particles in volume of the V for time intervals Δt. For this time according to (2.34) and (2.35) through surface of S from volume V leaves following a quantity of the particles:

$$N_1 = \oint_S \left[(Aw, n) - \frac{1}{2}B\frac{\partial w}{\partial n}\right] dS\Delta t.$$

A change in the number of particles within V produces change in concentration w(x, t). Let us count a change in the number of particles within V for the time Δt:

$$N_2 = \int_V [w(\mathbf{x},\ t + \Delta t) - w(\mathbf{x},\ t)]\, dV$$

or

$$N_2 = \int_V \frac{\partial w}{\partial t}\, dV \Delta t.$$

If within volume of the V not sources of corpuscular emission and does not occur their absorptions, then according to the law of conservation of matter $N_1 = N_2$, i.e.

$$\int_V \frac{\partial w}{\partial t}\, dV = -\oint_S \left[(\mathrm{A}w,\ \mathbf{n}) - \frac{1}{2}\, \mathrm{B} \frac{\partial w}{\partial n} \right] dS.$$

Applying the theorem of an Ostrogradskiy-Gauss, we will obtain

$$\int_V \frac{\partial w}{\partial t}\, dV = -\int_V \operatorname{div} \left(\mathrm{A}w - \frac{1}{2}\, \mathrm{B}\, \operatorname{grad} w \right) dV.$$

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Since volume V is arbitrary, then for any point must be implemented the equality

$$\frac{\partial w}{\partial t}+\operatorname{div}\left(Aw-\frac{1}{2}B\operatorname{grad}w\right)=0,$$

being nothing else but the equation of Fokker-Planck (2.27). The expression, which stands in the parenthesis, is the resulting particle flux through surface element of dS due to convection (Aw) and diffusions ($-\frac{1}{2}$ B grad w). Consequently, if we represent the equation of Fokker-Planck (2.27) in the divergent form

$$\frac{\partial w}{\partial t}+\operatorname{div}\Pi=0, \qquad (2.36)$$

then $\Pi(x, t)$ can be considered as the vector of the flow of probability density with the components

$$\Pi_i=A_iw-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(B_{ij}w),\ i=1,\ 2,\ \ldots,\ n. \qquad (2.37)$$

The equation of Fokker-Planck (2.36) expresses, thus, the differential law of conservation of probability.

Examples of the composition of the equations of Fokker-Planck. Using formulas (2.32) and (2.33) for coefficients A_i and B_{ij}, let us register the equation of Fokker-Planck for the probability density of

following error x(t) in the system with the integrator and the proportional-integrating filter (see an example in § 2.2). As was shown in § 2.2, for the system in question were possible the different methods of the introduction of the components of Markov process. If components x_1 and x_2 are introduced by the system of stochastic equations (2.21), then the equation of Fokker-Planck takes the form

$$\frac{\partial w(x_1, x_2, t)}{\partial t}+\frac{\partial}{\partial x_1}\left\{\left[x_2+\frac{1}{4}K^2n^2\frac{dN_0(x_1)}{dx_1}\right]w\right\}+$$
$$+\frac{\partial}{\partial x_2}\left\{\frac{1}{T}\left[-\left(1+KnT\frac{dF(x_1)}{dx_1}\right)x_2-KF(x_1)+T\frac{d^2\lambda}{dt^2}+\right.\right.$$
$$+\frac{d\lambda}{dt}+\frac{1}{4}K^2n\frac{d\left[\left(1-n-Kn^2T\frac{dF(x_1)}{dx_1}\right)\sqrt{N_0(x_1)}\right]}{dx_1}\times$$
$$\left.\left.\times\sqrt{N_0(x_1)}\right]w\right\}=\frac{1}{4}\frac{\partial^2}{\partial x_1^2}[n^2K^2N_0(x_1)w]+$$
$$+\frac{1}{2}\frac{\partial^2}{\partial x_1\partial x_2}\left[\frac{nK^2}{T}\left(1-n-Kn^2T\frac{dF(x_1)}{\partial x_1}\right)N_0(x_1)w\right]+$$
$$+\frac{1}{4}\frac{\partial^2}{\partial x_2^2}\left[\frac{K^2}{T^2}\left(1-n-Kn^2T\frac{dF(x_1)}{dx_1}\right)^2N_0(x_1)w\right], \quad (2.38)$$

where $x = x_1$.

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With the help of the second method of the introduction of the components of Markov process was obtained the system stochastic equations (2.26). To it corresponds the equation of Fokker-Planck for density $w(z_1, z_2, t)$:

$$\frac{\partial w}{\partial t}+z_2\frac{\partial w}{\partial z_1}+\frac{\partial}{\partial z_2}\left\{\left[-\frac{z_2}{T}-\frac{K}{T}F(x)+\frac{\lambda_1}{T}+\right.\right.$$
$$\left.\left.+\frac{1}{8}\frac{K^2n}{T}\frac{dN_0(x)}{dx}\right]w\right\}=\frac{1}{4}\frac{\partial^2}{\partial z_2^2}\left(\frac{K^2}{T^2}N_0(x)w\right), \quad (2.39)$$

moreover $x=z_1+T_1z_2$, $n=T_1/T$.

Fundamental solution. In order to find the solution of the equation of Fokker-Planck, it is necessary to, first of all, determine initial conditions. At the moment of time t=0 of the value of the Markov process of $x(0)=x_0$ they can be random with a probability density of $w_0(x)$. Then function $w_0(x)$ is initial condition for equation (2.27)

$$w(\mathbf{x}, 0)=w_0(\mathbf{x}). \quad (2.40)$$

If at zero time occurs the inclusion/connection of noise or ring closure of automatic control, then usually initial values are known accurately and are described by determined vector $\mathbf{x}_0=\{x_{01}, x_{02}, \ldots, x_{0n}\}$, in this case initial condition takes the form

$$w(\mathbf{x}, t)|_{t=0}=\delta(\mathbf{x}-\mathbf{x}_0)=\delta(x_1-x_{01})\delta(x_2-x_{02})\ldots\delta(x_n-x_{0n}). \quad (2.41)$$

The solution of the equation of Fokker-Planck, examined/considered in the unlimited phase space and which satisfies initial condition (2.41), is called the fundamental solution of the problem of Cauchy. As it follows from the definition, fundamental solution coincides with the probability density of transition w(x, t;

x_0).

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Fundamental solution makes it possible to determine the solution of the equation of Fokker-Planck with arbitrary initial condition (2.40)

$$w_1^*(\mathbf{x}, t) = \int_{\Xi} w_1^*(\mathbf{x}, t;\ \mathbf{x}_0)\, w_0(\mathbf{x}_0)\, d\mathbf{x}_0. \qquad (2.42)$$

Linear task. To find the unsteady solution of equation in the partial derivatives in the general case is very difficult. Exception/elimination compile an equation, which describe the behavior of linear systems. In these cases the coeffic ʦs of removal/drift in the equation of Fokker-Planck are the linear functions of the space coordinates $x_1, x_2, \ldots, x_n$:

$$A_i(\mathbf{x}, t) = \sum_{k=1}^{n} q_{ik}(t)\, x_k + r_i,$$

and the diffusion coefficients on coordinates x_k do not depend

$$B_{ij}(\mathbf{x}, t) = B_{ij}(t).$$

The system stochastic differential equations, which describe the behavior of linear system, is conveniently registered in the matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{Q}\mathbf{x} + \mathbf{R} + \mathbf{b}\Xi, \qquad (2.43)$$

where $\Xi = \|\xi_i^0\|$, $\mathbf{x} = \|x_i\|$, $\mathbf{R} = \|r_i\|$ — column vectors; $\mathbf{Q} = \|q_{ij}\|$, $\mathbf{b} = \|b_{ij}\|$ —

square matrices/dies.

It is possible to show that in linear system (2.43) vector x(t) is distributed according to normal law [19]

$$w(x,\ t;\ x_0)=\frac{1}{\sqrt{(2\pi)^n D}}\exp\left[-\frac{1}{2}(x-M)^+D^{-1}(x-M)\right], \tag{2.44}$$

where $D=\|\overline{(x_i-m_i)(x_j-m_j)}\|$ — mutual correlation matrix/die; D - determinant of matrix/die D; x_0 - vector of initial conditions; $M=\|m_i\|$ — vector of average/mean values. Symbol $^+$ indicates the transposition of matrix/die.

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Matrices/dies M and D are found as a result of solving the equations

$$\frac{dM}{dt}=QM+R,\ M(0)=x_0,$$
$$\frac{dD}{dt}=QD+DQ^+ +\frac{1}{2}bb^+,\ D(0)=D_0. \tag{2.45}$$

However, the use/application of equations of Fokker-Planck for the analysis of linear tasks is not significant, since result (2.44) can be obtained by the methods of the correlation theory of random processes [10].

The practical value of the equation of Fokker-Planck they

acquire during the research of nonlinear control systems and, in particular, during the analysis of the disruption/separation of tracking.

During the research of the disruption/separation of tracking the Markov process x(t) takes values not on the entire infinite plane, but on certain of its part; therefore the equation of Fokker-Planck must be supplemented by boundary conditions. To the discussion of boundary conditions for the tasks about the first reaching/achievement of boundary by multidimensional Markov process is dedicated § 2.5.

2.4. Possibilities of simplification in the equation of Fokker-Planck.

In certain cases the equation of Fokker-Planck can be given to the simpler form by replacing the variable/alternating of differentiation. Let us pause at the most known methods of replacement.

Replacement of V. Feller's variable/alternating [40]. With the help of the introduction of the new space coordinate

$$x_1 = \int_0^x \frac{d\zeta}{\sqrt{B(\zeta)}} \qquad (2.46)$$

it is possible to reduce the equation of Fokker-Planck

$$\frac{\partial w(x, t)}{\partial t}+\frac{\partial}{\partial x}[A(x, t)w]=\frac{1}{2}\frac{\partial^2}{\partial x^2}[B(x)w] \quad (2.47)$$

to the equation with the constant coefficient of the diffusion

$$\frac{\partial w_1(x_1, t)}{\partial t}+\frac{\partial}{\partial x_1}[A_1(x_1, t)w_1]=\frac{1}{2}\frac{\partial^2 w_1}{\partial x_1^2}, \quad (2.48)$$

where $w_1(x_1, t)=\sqrt{B(x)}w(x, t)$.

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In this case the coefficient of removal/drift takes the form

$$A_1(x_1, t)=\frac{1}{\sqrt{B(x)}}\left[A(x, t)-\frac{1}{4}\frac{dB(x)}{dx}\right].$$

In this expression the variable/alternating x must be substituted according to (2.46) the new variable/alternating x_1.

Feller's ideas can be used for simplification in one special case of the equation of the second order, which describes the system of self-alignment with the integrator and the integrating filter:

$$\frac{\partial w(x, y, t)}{\partial t}+y\frac{\partial w}{\partial x}+\frac{\partial}{\partial y}[A(x, y, t)w]=\frac{B(x)}{2}\frac{\partial^2 w}{\partial y^2}. \quad (2.49)$$

F equation (2.49) it is characteristic that coefficient of diffusion B(x) depends only on following error x and does not depend on the derivative y. The replacement of the variable/alternating

$$x_1=\int_0^x \frac{d\zeta}{\sqrt{B(\zeta)}},\ y_1=\frac{y}{\sqrt{B(x)}} \tag{2.50}$$

reduces equation (2.49) to the form

$$\frac{\partial w_1(x_1,\ y_1,\ t)}{\partial t}+y_1\frac{\partial w_1}{\partial x_1}+\frac{\partial}{\partial y_1}[A_1(x_1,\ y_1)w]=\frac{1}{2}\frac{\partial^2 w_1}{\partial y_1^2}, \tag{2.51}$$

where

$$A_1(x_1,\ y_1,\ t)=\frac{A(x,\ y,\ t)-\frac{1}{2}y_1^2B'(x)}{\sqrt{B(x)}},$$

$$w_1(x_1,\ y_1,\ t)=B(x)w(x,\ y,\ t).$$

The replacement of I. D. Cherkasov's variable/alternating [43] makes it possible to reduce the one-dimensional equation of Fokker-Planck

$$\frac{\partial w(x,\ t)}{\partial t}+\frac{\partial}{\partial x}[A(x,\ t)w]=\frac{1}{2}\frac{\partial^2}{\partial x^2}[B(x,\ t)w] \tag{2.52}$$

to the simplest form of the equation of the thermal conductivity

$$\frac{\partial w_1(x_1,\ t_1)}{\partial t_1}=\frac{1}{2}\frac{\partial^2 w_1(x_1,\ t_1)}{\partial x_1^2}, \tag{2.53}$$

fundamental solution of which is well known

$$w_1(x_1,\ t_1)=\frac{1}{\sqrt{2\pi t_1}}\exp\left[-\frac{(x_1-x_{10})^2}{2t_1}\right].$$

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This conversion is possible, if it is converted into zero determinant

$$\begin{vmatrix} \alpha(x, t) & \beta(x, t) & \gamma(x, t) \\ \alpha'_x(x, t) & \beta'_x(x, t) & \gamma'_x(x, t) \\ \alpha''_{xx}(x, t) & \beta''_{xx}(x, t) & \gamma''_{xx}(x, t) \end{vmatrix} = 0, \quad (2.54)$$

where

$$\alpha(x, t) = \sqrt{B(x, t)} \int_0^x [B(\zeta, t)]^{-1/2} d\zeta, \quad \beta(x, t) = \sqrt{B(x, t)},$$

$$\gamma(x, t) = 2A(x, t) - \frac{1}{2} B'_x(x, t) -$$

$$- \sqrt{B(x, t)} \int_0^x B'_t(\zeta, t) [B(\zeta, t)]^{-3/2} d\zeta.$$

In the resulting expressions the index indicates the variable/alternating, in terms of which is produced the differentiation.

With satisfaction of condition (2.54) new variable/alternating are determined by the expressions

$$t_1 = \int_0^t \exp[-2D(x, \tau)] d\tau, \quad x_1 = \frac{\alpha(x, t)}{\beta(x, t)} \exp[-D(x, t)] +$$

$$+ \frac{1}{2} \int_0^t \frac{\alpha(x, \tau)\gamma'_x(x, \tau) - \gamma(x, \tau)\alpha'_x(x, \tau)}{\beta(x, \tau)} \exp[-D(x, \tau)] d\tau,$$

where

$$D(x, t) = \frac{1}{2} \int_0^t \left[\gamma'_x(x, \tau) - \gamma(x, \tau) \frac{\beta'_x(x, \tau)}{\beta(x, \tau)} \right] d\tau.$$

Function $w_1(x_1, t_1)$ is connected with the initial density $w(x, t)$ with the relationship/ratio

$$w(x, t) = \left| \frac{\partial x_1}{\partial x} \right| w_1(x_1, t_1).$$

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In the particular case when B(x, t)=B(x), initial equation it is possible to reduce to form (2.53), if the coefficient of removal/drift is determined by the expression

$$A(x, t)=\beta(x)\left[C(t)\int_{0}^{x}\frac{d\xi}{\beta(\xi)}+\frac{1}{2}\beta'_{x}(x)+C_{1}(t)\right],$$

where C(t) and $C_1(t)$ the arbitrary functions of time. Hence it follows that with B(x, t)=B=const the reduction of the equation of Fokker-Planck to form (2.53) is possible only in such a case, when the coefficient of removal/drift is the linear function x:

$$A(x, t)=xC(t)+C_1(t).$$

This occurs only in the linear regulating circuits. Thus, the limitations, superimposed on the form of the function A(x, t) and B(x, t), prove to be very rigid.

Replacement of V. L. Lebedev's variable/alternating. In work [31] is proposed the following method of replacing the variable/alternating: $x_1=\psi(x, t)$, $t_1=\varphi(t)$, that makes it possible to reduce equation (2.52) to substantially the simpler form

$$\frac{\partial w_1(x_1, t_1)}{\partial t_1}=\Phi(t_1)\left\{-\frac{\partial}{\partial x_1}[A^*(x_1)w_1]+\frac{1}{2}\frac{\partial^2 w_1}{\partial x_1^2}\right\}. \quad (2.55)$$

After computing partial derivatives in equation (2.52) and

taking into account that in the modified equation the diffusion coefficient must be equal to $\Phi(t_1)$, we will obtain the following expression for the function ψ:

$$\psi(x, t) = \sqrt{\varphi'(t)\Phi(t_1)} \int_0^x \frac{dy}{\sqrt{B(y, t)}} + C(t). \qquad (2.56)$$

where C(t) - the arbitrary function of time.

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In this case the coefficient of removal/drift is equal to

$$A_1(x_1, t_1) = \frac{1}{2\sqrt{\varphi'(t)}} \{M(t) + [N(t)U(t) + $$
$$+ 2U'(t)]\alpha(x, t) + U(t)\beta(x, t)\}, \qquad (2.57)$$

where

$$\alpha(x, t) = \int_0^x [B(\zeta, t)]^{-1/2} d\zeta;$$

$$\beta(x, t) = \frac{1}{\sqrt{B(x, t)}}[2A(x, t) - B'_x(x, t)] -$$
$$- \int_0^x \frac{B'_t(\zeta, t)}{[B(\zeta, t)]^{3/2}} d\zeta;$$

$$M(t) = \frac{2}{\sqrt{\varphi'(t)}} \frac{dC}{dt}, \; N(t) = \frac{\varphi''(t)}{\varphi'(t)}, \; U(t) = \sqrt{\Phi(t_1)}.$$

The entering expression (2.57) functions $\alpha(x, t)$ and $\beta(x, t)$ are uniquely determined by coefficients of A and B of the initial equation of Fokker-Planck (2.52), and functions M(t), N(t) and U(t) can be varied. They are chosen so that the coefficient of the removal/drift of the modified equation would take the form

$$A_1(x_1, t_1) = \Phi(t_1)A^*(x_1). \qquad (2.58)$$

Representation (2.58) is possible upon the satisfaction of the following equation:

$$\left[U(t)\frac{\partial A_1(x_1, t_1)}{\partial t}-2A_1(x_1, t_1)\frac{dU(t)}{dt}\right]\frac{\partial\psi(x, t)}{\partial x}= \\ =U(t)\frac{\partial A_1(x_1, t_1)}{\partial x}\frac{\partial\psi(x, t)}{\partial t}. \qquad (2.59)$$

Entering this equation derivatives of function $\psi(x, t)$ can be calculated as follows:

$$\frac{\partial\psi}{\partial x}=U(t)\sqrt{\frac{\varphi'(t)}{B(x, t)}},$$

$$\frac{\partial\psi}{\partial t}=\frac{\sqrt{\varphi'(t)}}{2}\{M(t)+[N(t)U(t)+2U'(t)]\alpha(x, t)- \\ -2U(t)\alpha'_t(x, t)\}. \qquad (2.60)$$

After substituting expressions (2.57) and (2.60) in (2.59), we will obtain ordinary differential equation for the unknown function N(t), into which enter also the arbitrary functions M(t) and U(t). Them are chosen so that the obtained equation would have a solution (at least trivial). Then the initial equation of Fokker-Planck (2.52) it is possible to reduce to form (2.55).

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After determining function N(t), we find

$$t_1=\varphi(t)=\int_0^t\exp\left[\int_0^u N(v)\,dv\right]du. \qquad (2.61)$$

Entering in (2.56) function C(t) is determined by the expression

$$C(t)=\frac{1}{2}\int_0^t M(u)\exp\left[\int_0^u N(v)\,dv\right]du. \qquad (2.62)$$

2.5. Boundary conditions in the tasks about the disruption/separation of tracking.

In the tasks about the disruption/separation the following error x(t) is or the component of the multidimensional Markov process x(t) [for example, (2.19)], or the linear combination of components [for example, (2.26)]. By disruption/separation of tracking, as was noted in S 1.3, frequently is understood the first output of the trajectory of random process x(t) beyond the established/installed boundaries γ_1, γ_2, usually connected with the aperture of the discriminatory characteristic F(x). Therefore those realizations of Markov process in which value x at certain moment of time τ falls outside boundaries γ_1, γ_2, must be withdrawn from the examination with $t>\tau$. For this on the straight lines $x=\gamma_1$ and $x=\gamma_2$ are placed the absorbing boundaries.

Mathematical recording of boundary conditions. Let us switch over to the mathematical description of boundary conditions for the equation of Fokker-Planck in the tasks about the first reaching/achievement of boundaries by the multidimensional Markov

process of x(t). Let in the n-dimensional region Ω with boundary of G assignedly is the equation of Fokker-Planck

$$\frac{\partial w}{\partial t}+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}[A_i(\mathrm{x},\ t)\,w]=\frac{1}{2}\sum_{i,\ j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}[B_{ij}(\mathrm{x},\ t)\,w], \quad (2.27)$$

the matrix/die of diffusion $\|B_{ij}\|$ can be degenerate.

Boundary conditions of G must be such that into the region Ω of phase space would not be allowed/assumed trajectories from without. In the one-dimensional case for this it suffices to require

$$w(x,\ t)|_{x\in G}=0. \quad (2.63)$$

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For the multidimensional tasks condition (2.63) can prove to be too rigid [86, 87], since it removes not only the entering, but also outgoing from the region Ω trajectories. In order to find sufficient and necessary boundary conditions, let us isolate on surface of G the regular part $\widetilde{G}$, on which are in principle possible the trajectories, entering the region Ω. In order to reduce them, let us require

$$w(\mathrm{x},\ t)|_{x\in\widetilde{G}}=0. \quad (2.64)$$

Since through the remaining part of boundary G-$\widetilde{G}$ trajectories cannot return to the region Ω, then it is sufficient so that w(x, t) on G-$\widetilde{G}$ would satisfy only the equation of Fokker-Planck. Required satisfaction of any further conditions should not be.

Let us consider the method of the isolation/liberation of the regular part of the boundary.

The points of boundary x belongs $\tilde{G}$, if is implemented one of following two conditions [34, 41]:

1. The matrix/die of diffusion is not degenerated in direction n, normal to the boundary

$$\sum_{i,j=1}^{n} B_{ij}(x)\, n_i n_j \neq 0, \qquad (2.65)$$

where n_i— direction cosines of the external standard/normal n.

2. Matrix/die of diffusion B is degenerated in direction n, but is satisfied condition

$$\sum_{i=1}^{n}\left[A_i(x)-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial B_{ij}(x)}{\partial x_j}\right]n_i<0. \qquad (2.66)$$

Physical treatment. Let us clarify the formulated conditions. In such a case, when matrix/die B is degenerated in direction n, is satisfied the condition

$$\sum_{i,j=1}^{n} B_{ij}(x)\, n_i n_j = 0. \qquad (2.67)$$

Taking into account that elements/cells B_{ij} are connected with the coefficients stochastic equations b_{ij} with dependence (2.32),

expression (2.67) can be registered in the form

$$\sum_{i=1}^{n} b_{ij}(\mathrm{x})\, n_i = 0,\ j = 1,\ 2, \ldots, n. \qquad (2.68)$$

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Latter/last equalities mean that external noise effect of the type of white noise is absent from the direction normal to the boundary. Therefore normal to the boundary of the component of process x(t) is differentiated, i.e., sufficiently smooth. This makes it possible unambiguously to determine, in what direction moves the trajectory near the boundary - to it or from it. Actually/really, taking into account (2.37), let us consider the component normal to the boundary of the vector of the flow

$$(\Pi, \mathrm{n}) = \sum_{i=1}^{n} \left[A_i(\mathrm{x})\, w - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial (B_{ij}(\mathrm{x})\, w)}{\partial x_j} \right] n_i.$$

Taking into account (2.68), after simple conversions we will obtain

$$(\Pi, \mathrm{n}) = \sum_{i=1}^{n} \left[A_i(\mathrm{x}) - \frac{1}{2} \sum_{j=1}^{n} \frac{\partial B_{ij}(\mathrm{x})}{\partial x_j} \right] w(\mathrm{x}, t)\, n_i, \qquad (2.69)$$

If normal component of flow (Π, n) at point $x \in G$ is positive, then through this point in the trajectory they leave from the region Ω. In this case of x does not belong $\tilde{G}$. To enter into the region Ω trajectories can only through those sections of the boundaries on which (Π, n) <0. Using a property of probability density $w(x, t) \geqslant 0$,

from (2.69) we will obtain the second condition of the accessory/affiliation of point x with the regular part of the boundary.

If condition (2.68) is not satisfied, then the component of Markov process, normal to the boundary, is nondifferentiated. Trajectory x(t), approaching the boundary, manages a countless multitude of times to cross it. Therefore in order to ensure the condition for absorption on the boundary, in such situation it is necessary to require satisfaction of condition (2.64).

In the majority of the tasks about the disruption/separation of tracking the boundary of the region of tracking is normal to one of the coordinate axes x_i of phase space. In this case satisfaction of condition (2.68) is equivalent so that in the equation of Fokker-Planck is absent the second density derivative of probability in terms of variable/alternating x_i. In this case to value w(x, t) on coordinate x_i can be superimposed less than limitations.

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It is here convenient to lead analogy with the ordinary differential equations: for the unique solution of second order equation it is necessary to be given two boundary conditions, but during its

degeneration into the first-order equation the presence of two limitations to the unknown function can lead generally to the absence of solution.

Equation (2.27), initial condition (2.40) and boundary conditions (2.64) form boundary-value problem for the equation of Fokker-Flanck. From the results of works [34, 41] it follows that the solution of the boundary-value problem presented exists and it is singular. Since boundary condition (2.64) is assigned only on the regular part of $\tilde{G}$ of boundary, then on the remaining part of $G-\tilde{G}$ of boundary probability density is determined in the course of solution of task.

Let us consider several examples to the recording of boundary conditions in different tasks.

Example 1. Control system with the integrator and the integrating filter is described stochastic equations:

$$\left.\begin{aligned} & x_1 = x, \\ & \frac{dx_1}{dt} = x_2, \\ & \frac{dx_2}{dt} = -\frac{KF(x_1) + x_2}{T} + \frac{d^2\lambda}{dt^2} + \frac{1}{T}\frac{d\lambda}{dt} + \\ & \qquad + \frac{K}{T}\sqrt{N_0(x_1)}\,\xi^0(t), \end{aligned}\right\} \tag{2.70}$$

which are obtained from (2.21) with $n=T_1/T=0$. The corresponding

equation of Fokker-Planck is ultra-parabolic and takes the form

$$\frac{\partial w(x_1, x_2, t)}{\partial t}+\frac{\partial}{\partial x_1}(A_1 w)+\frac{\partial}{\partial x_2}(A_2 w)=\frac{1}{2}\frac{\partial^2}{\partial x_2^2}(B_{22} w), \tag{2.71}$$

where

$$A_1=x_2,\ A_2=-\frac{KF(x_1)+x_2}{T}+\frac{d^2\lambda}{dt^2}+\frac{1}{T}\frac{d\lambda}{dt},$$
$$B_{11}=B_{12}=B_{21}=0,\ B_{22}=\frac{K^2 N_0(x_1)}{2T^2}. \tag{2.72}$$

In this example of formula for the coefficients of removal/drift (2.31), (2.33) they coincide, since noise functions only on one coordinate x_2, but spectral density $N_0(x_1)$ depends on following error x_1 and does not depend on its derivative x_2.

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The region of tracking Ω on the phase plane (x_1, x_2) is limited by lines $x_1=\gamma_1$ and $x_1=\gamma_2$ (Fig. 2.2), which form boundary of G. On the left side of boundary $(x_1=\gamma_1)$ the direction cosines of external standard/normal $n=\{n_1, n_2\}$ are equal to $n_1=-1$, $n_2=0$. On the right side of boundary $(x_1=\gamma_2)$ $n_1=1$, $n_2=0$. On the entire boundary of G the matrix/die of diffusion B is degenerated, since condition (2.68) is satisfied both on the left and on the right sides of the boundary. Let us isolate the regular part of boundary $\tilde{G}$. By virtue of (2.66) for points $x \in \tilde{G}$ must be satisfied the condition

$$\left\{A_1(x)-\frac{1}{2}\left[\frac{\partial B_{11}(x)}{\partial x_1}+\frac{\partial B_{12}(x)}{\partial x_2}\right]\right\}n_1+$$
$$+\left\{A_2(x)-\frac{1}{2}\left[\frac{\partial B_{12}(x)}{\partial x_1}+\frac{\partial B_{22}(x)}{\partial x_2}\right]\right\}n_2<0. \quad (2.73)$$

Using relationship/ratio (2.72) and taking into account that $n_2=0$, we convert (2.73) to the following form:

$$x_2 n_1<0. \quad (2.74)$$

With $x_1=\gamma_1$ the direction cosine $n_1=-1$; therefore condition (2.74) is implemented with $x_2>0$. On right boundary ($x_1=\gamma_2$) $n_1=1$; therefore condition (2.74) is correct with $x_2<0$. Thus, the regular part of boundary G form rays/beams $x_1=\gamma_1$, $0<x_2<\infty$ and $x_1=\gamma_2$, $-\infty<x_2<0$, on which is assigned the condition for absorption (2.64).

Let us clarify this example. From equations (2.70), which describe the two-dimensional Markov process of $x(t)$, it follows that $x_2(t)$ - the continuous random process, undifferentiable not in a moment of time.

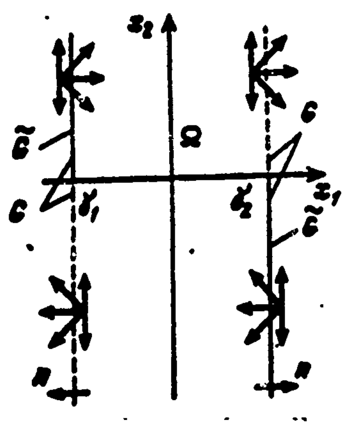

Fig. 2.2. Boundary conditions in the system of the second order with the integrating filter.

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On component $x_1(t)$ random jerks/impulses of the type of white noise directly do not act, since

$$x_1(t)=\int_0^t x_2(\tau)\,d\tau+x_1(0).$$

Therefore process $x_1(t)$ is smoother than $x_2(t)$. At each moment of time it has the final derivative $dx_1(t)/dt=x_2(t)$. With $x_2>0$ the motion of trajectories on the phase plane occurs only in the direction of an increase in coordinate x_1, when $x_2<0$ - in the opposite direction. The directions of the motion of phase trajectories are shown in Fig. 2.2 by arrows/pointers. To enter into the region Ω phase trajectories can only through the regular part of

boundary $\tilde{G}$. In order not to allow this, it is necessary to require $w(x, t)|_{x\in\tilde{G}}=0$. Let us note that if we assume w(x, t)=0 on the entire boundary of G, then there does not exist the nontrivial solution of equation (2.71) [86].

Example 2. The system of self-alignment with the integrator and the proportional-integrating filter is described, as shown in § 2.2, by two methods. Using the first method is introduced Markov process $x(t)=(x_1(t), x_2(t))$, controlled by the system stochastic equations (2.21). The corresponding equation of Fokker-Planck takes form (2.38). As it follows from (2.21), the matrix/die of the intensities of the white noises b in this example has the following components:

$$b_{11}=nK\sqrt{N_0(x_1)}, \quad b_{12}=b_{22}=0,$$
$$b_{21}=K\left(1-n-Kn^2T\frac{dF(x_1)}{dx_1}\right)\sqrt{N_0(x_1)}.$$

The region of tracking Ω is the same as in previous example $\gamma_1\geqslant x_1\geqslant\gamma_2, -\infty<x_2<\infty$). But the matrix/die of diffusion not at one point of boundary is degenerated, since

$$\sum_{k=1}^{n} b_{1k}n_k=b_{11}n_1+b_{21}n_2=b_{11}n_1=\pm b_{11}\neq 0.$$

Therefore entire/all boundary is regular $\tilde{G}\equiv G$, and the condition for absorption (2.64) must be assigned on the entire boundary

$$w(x, t)|_{x\in G}=0. \qquad (2.75)$$

Let us clarify this result. In contrast to previous example both

components $x_1(t)$ and $x_2(t)$ two-dimensional Markov process are undifferentiable.

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Consequently, phase trajectories on the plane (x_1, x_2) are strongly cut and if we designate the possible directions of the motion of trajectories by arrows/pointers, then it should be directed them in different directions independent of the quadrant of plane (x_1, x_2). Therefore in order not to allow the return of trajectories to the region Ω, it is necessary to require satisfaction of condition (2.75) on the entire boundary of G.

With the help of the second method of the introduction of multidimensional Markov process is obtained the system stochastic equations (2.26), which describes Markov process $z(t)=\{z_1(t), z_2(t)\}$. Following error x(t) is connected with components z_1 and z_2 with the relationship/ratio

$$x=z_1+T_1z_2.$$

The region of tracking Ω has a boundary of G (Fig. 2.3), formed by the lines

$$z_1+T_1z_2=\gamma_1,\quad z_1+T_1z_2=\gamma_2.$$

In spite of the fact that noise $\xi(t)$ enters only in one equation of system (2.26), condition (2.68) of degenerating the matrix/die of

diffusion B in the direction, perpendicular to boundary, is not satisfied. Actually/really, in the case

$$b_{11}=b_{12}=b_{22}=0,\ b_{21}=\frac{K}{T}\sqrt{N_0(x)},$$

in question therefore condition (2.68) takes the form

$$b_{21}n_2=0. \tag{2.76}$$

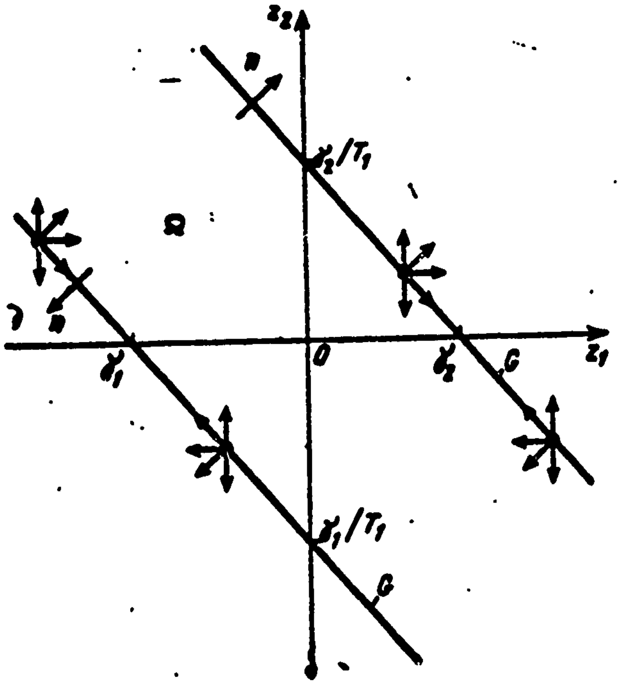

Fig. 2.3. Boundary conditions in the system of the second order with proportional-integrating filter.

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The direction cosines of external normals to the boundaries are equal to

$$n_1 = \pm \frac{1}{\sqrt{1+T_1^2\left(\frac{e_2}{e_1}\right)^2}}; \quad n_2 = \pm \frac{T_1 \frac{e_2}{e_1}}{\sqrt{1+T_1^2\left(\frac{e_2}{e_1}\right)^2}},$$

where e_1 and e_2 - scale factors along the axes z_1 and z_2. Hence it is apparent that with $T_1>0$ condition (2.76) is satisfied not at one

point of boundary G. Thus, entire/all boundary G in this example is regular.

Let us clarify the obtained result. In form the system stochastic equations (2.26), which describes two-dimensional Markov process (z_1, z_2), coincides with system (2.70). Therefore in this case, just as in example 1, phase trajectories are smooth smooth curves. Difference lies in the fact that in a latter/last example boundary G is nonorthogonal to axis z_1. Therefore at each point of boundary are possible both the outgoing from the region Ω trajectories and entering it (Fig. 2.3). The trajectory which at certain moment of time t for the first time left abroad of G, at the following moment/torque can return conversely. In order not to allow this, it is necessary to require satisfaction of condition (2.64) on the entire boundary of G.

2.6. Boundary-value problem for the equation of Pontriagin.

Equation of Pontriagin. Probability density w(x, t), obtained as a result of solving the boundary-value problem for the equation of Fokker-Planck (2.27), makes it possible to determine the probability of the first reaching/achievement of boundaries of the region Ω:

$$P_i(t) = 1 - \int_{\Omega} w(\mathbf{x}, t)\, d\mathbf{x}. \qquad (2.77)$$

If at zero time t=0 value x is known accurate

$$w(\mathbf{x}, 0) = \delta(\mathbf{x} - \mathbf{x}_0),$$

then for probability $P(x_0, t)$ of the first reaching/achievement of boundary is correct the equation of Pontriagin [35]

$$\frac{\partial P}{\partial t}=\sum_{i=1}^{n} A_i(x_0, t)\frac{\partial P}{\partial x_{0i}}+\frac{1}{2}\sum_{i,j=1}^{n} B_{ij}(x_0, t)\frac{\partial^2 P}{\partial x_{0i}\partial x_{0j}}. \qquad (2.78)$$

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Function $P(x_0, t)$ is probability that for a period of time t the representative point at least one time fall outside the limits of region Ω, being found at zero time at point with coordinates $x_0=(x_{01}, x_{02}, x_{03}, \ldots, x_{0n})$ within the region. Coefficients A_i and B_{ij} equation (2.78) make the same sense, as in the equation of Fokker-Planck (2.27).

If the initial state of dynamic system with the distribution

$$w(x, 0)=w_0(x),$$

then probability P(t) of the first reaching/achievement of boundaries is randomly determined by the expression

$$P(t)=\int_{\Omega} P(x_0, t)\, w_0(x_0)\, dx_0. \qquad (2.79)$$

Boundary conditions. For the unique solution of equation (2.78) it is necessary to formulate initial and boundary conditions.

If at zero time $t_0=0$ representative point $x=x_0$ is found within the region Ω, then probability that at the same moment of time the trajectory fall outside the limits of region Ω, it is equal to zero:

$$P(x_0, 0)=0. \qquad (2.80)$$

Initial condition (2.80) means that the phase trajectory cannot for infinitesimal time pass the final distance, which separates/liberates point x_0 from the boundary.

Boundary conditions for the equation of Pontriagin are assigned on the regular part G* of boundary G and are written/recorded in the form

$$P(x_0, t)|_{x_0 \in G^*}=1. \qquad (2.81)$$

Condition (2.81) characterizes the authenticity of the emergence of trajectory from phase field Ω, if trajectory is found on G*.

Of the regular part of boundary G* for the equation of Pontriagin it is determined as follows. Point x_0 lies/rests on the regular part of boundary G*, if is implemented one of the conditions:

1. The matrix/die of diffusion B is not degenerated in direction n, normal to the boundary

$$\sum_{i,j=1}^{n} B_{ij}(x_0, t)\, n_i n_j \neq 0. \qquad (2.82)$$

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2. Matrix/die B is degenerated, but is fulfilled inequality

$$\sum_{i=1}^{n}\left[A_i(x_0, t)-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial B_{ij}(x_0, t)}{\partial x_{0j}}\right]n_i>0. \quad (2.83)$$

Comparing the enumerated conditions with the determination of the regular part of the boundary for the equation of Fokker-Planck, let us note the coincidence of conditions (2.82) and (2.65) and the contrast of conditions (2.83) and (2.66). A difference in conditions (2.83) and (2.66) is caused by different physical sense of three-dimensional/space variable/alternating in the equations of Fokker-Planck and Pontriagin. In the equation of Fokker-Planck these variable/alternating are connected with the current following error, while in the equation of Pontriagin - with the initial state of system. On the parts of the boundary where matrix/die B is degenerated, it is possible to unambiguously indicate the direction of phase trajectories. Trajectory, which is located at the moment of time t on the boundary, leaves at the following moment/torque region Ω only in such a case, when normal to the boundary component of the flow of probability is positive, i.e., is satisfied condition (2.83).

When the matrix/die of diffusion is not degenerated, the phase trajectory is nondifferentiated. Reaching boundary of G, trajectory exceeds the limits of region Ω independent of flow direction.

For the one-dimensional equation of Pontriagin boundary conditions take the following form:

$$P_1(x_0, t)|_{x_0=\gamma_1} = P(x_0, t)_{x_0=\gamma_2} = 1. \qquad (2.84)$$

Equation (2.78), initial condition (2.80) and boundary condition (2.81) form boundary-value problem for the equation of Pontriagin.

Example. Let us compose boundary-value problem for determining the probability of disrupting/separating the tracking in the regulating circuit with the integrator and the integrating filter in the feedback loop (see an example in S 2.5).

The behavior of the system in question is described stochastic equations (2.70). Hence it follows that the equation of Pontriagin takes the form

$$\frac{\partial P(x_{01}, x_{02}, t)}{\partial t} = A_1 \frac{\partial P}{\partial x_{01}} + A_2 \frac{\partial P}{\partial x_{02}} + \frac{B_{22}}{2} \frac{\partial^2 P}{\partial x_{02}^2}, \qquad (2.85)$$

where coefficients A_i and B_{ij} are determined by expressions (2.72).

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By disruption/separation of tracking is understood the first output of process x(t) for the level γ_1 or γ_2, moreover $\gamma_1 < x_0 < \gamma_2$. The

region of tracking Ω on the phase plane (x_{e1}, x_{e2}) is limited by lines $x_{e1}=\gamma_1$ and $x_{e1}=\gamma_2$ (see Fig. 2.2).

In accordance with (2.80) initial condition for the boundary-value problem in question is written/recorded in the form

$$P(x_{e1}, x_{e2}, 0) = 0.$$

Let us register boundary conditions. The matrix/die of the diffusion coefficients in this example is degenerated, and condition (2.83) is equivalent to the following:

$$x_{e2} n_1 > 0, \tag{2.86}$$

where n_1 — direction cosine of external normal to the boundary, equal to

$$n_1 = \begin{cases} -1 \overset{(1)}{\text{при}}\ x_{e1} = \gamma_1, \\ \ \ 1 \ \text{при}\ x_{e1} = \gamma_2. \end{cases}$$

Key: (1). with.

Thus, condition (2.86) is satisfied in the following sections of boundary:

$$x_{e1} = \gamma_1 \ \overset{(1)}{\text{при}}\ x_{e2} < 0$$

and

$$x_{e1} = \gamma_2 \ \overset{(1)}{\text{при}}\ x_{e2} > 0.$$

Key: (1). with.

These straight lines are shown in Fig. 2.2 by dotted line. They

form the regular part G* of boundary G for the boundary-value problem of the equation of Pontriagin.

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Chapter 3.

DISRUPTION OF TRACKING IN QUASI-STATIONARY SYSTEMS.

The systems of tracking, subjected to disruption/separation, are in principle unsteady. In this chapter are analyzed the systems in which up to the moment/torque of the beginning of observation had time to be completed all transient processes, and dynamic following error was constant during entire mode/conditions of tracking. Such systems of automatic tracking let us name quasi-stationary.

3.1. Application of the theory of the ejections of random processes.

Poisson's law. In many radio engineering tasks the disruption/separation of tracking can be considered as the output of following error $x(t)$ beyond the limits of some fixed levels γ_1, γ_2, connected one way or another with the aperture of the discriminatory characteristic $F(x)$. This makes it possible to use for the analysis

of disruption/separation some positions of the correlation theory of ejections [17].

It is known that the distribution of the ejections of the fluctuations above the threshold γ, which noticeably exceeds actual stress of fluctuations ($\gamma \gg \sigma_x$), obeys the law of Poisson

$$P(n, t_u) = \frac{(\nu t_u)^n}{n!} e^{-\nu t_u}, \qquad (3.1)$$

where $P(n, t_u)$ - probability of appearance for time t_u is exact n of ejections; ν - frequency of ejections by which is understood an average number of intersections with the process of determined by the sign of derivative x(t) of the level γ per unit time.

On the basis (3.1) the probability of the appearance at least of one ejection of noise above the level γ for time t_u is determined by the formula

$$P(t_u) = 1 - P(0, t_u) = 1 - e^{-\nu t_u}. \qquad (3.2)$$

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Identifying disruption/separation with the reaching/achievement by the following error x(t) of one of the boundaries of the aperture of discriminator γ_1 or γ_2 and taking into account that reaching/achievement of right or left boundary on a comparatively small noise level is events mutually independent, we will obtain

$$P(t_u) = P_1(t_u) + P_2(t_u) - P_1(t_u) P_2(t_u), \qquad (3.3)$$

where $P_{1(2)}(t_n)=1-\exp[-\nu_{1(2)}t_n]$ — probability of reaching/achievement by process of x(t) of boundaries γ_1 and γ_2 corresponding ν_1, ν_2 - frequencies of the ejections of error x(t) for the levels γ_1 and γ_2.

For small probabilities of disruption/separation $P<0,1\div0,2$, which are of special interest in the applications/appendices, instead of (3.3) it is possible to register

$$P(t_n)=1-e^{-(\nu_1+\nu_2)t_n}\approx(\nu_1+\nu_2)t_n. \qquad (3.4)$$

Thus, the calculation of the probability of disrupting/separating the tracking when making these assumptions is reduced to the definition of the frequencies of the ejections ν_1 and ν_2 whose sum can be considered as the frequency of disruptions/separations.

The frequency of the ejections of the random differentiated process x(t) above the fixed level γ is determined by following formula [17]:

$$\nu(\gamma)=\int_0^\infty \dot{x}w(\gamma,\dot{x})\,d\dot{x},$$

where $w(\gamma,\dot{x})=w(x,\dot{x})|_{x=\gamma}$ - two-dimensional density of distribution of process and it by derivative, undertaken with $x=\gamma$.

However, to calculate the frequency of ejections from the given formula in the general case is difficult, since it is necessary to know two-dimensional probability density $w(x, \dot{x})$. Exception is the normal stationary process of x(t), for which $w(x, \dot{x})$ is equal to the product of one-dimensional densities and can be comparatively easily determined.

Frequency of ejections in the linear system. Let us consider the system, which has in the limits of aperture the linear characteristic of discriminator (Fig. 3.1). Let us assume also, that spectral density N_0 of normal noise $\xi(t)$, which led to the output discriminator, does not depend on disagreement/mismatch x. In such systems process x(t) up to the moment of separation is developed in the linear section of discriminatory characteristic; therefore during the determination of the frequency of ejections system can be considered linear.

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In the linear system the following error x(t) is distributed according to the normal law

$$w(x, t)=\frac{1}{\sqrt{2\pi}\sigma_x(t)}\exp\left\{-\frac{(x-m_x(t))^2}{2\sigma_x^2(t)}\right\}, \qquad (3.5)$$

where $\sigma_x^2(t)$, $m_x(t)$ - dispersion and the mathematical expectation of

process.

If the normal process x(t) is stationary and central ($\sigma_x^2(t)=\sigma_x^2$ and $m_x(t)=0$) and has twice differentiated correlation function $r(\tau)=\sigma_x^2 R(\tau)$, then the frequency of the ejections of this process for the level γ is determined [2, 17] according to the formula

$$\nu=\frac{1}{2\pi}\sqrt{-R''(0)}e^{-\gamma^2/2\sigma_x^2}, \tag{3.6}$$

where

$$R''(0)=\left.\frac{d^2R(\tau)}{d\tau^2}\right|_{\tau=0}.$$

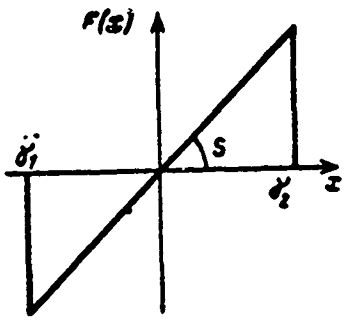

Fig. 3.1. Characteristic of "linear" discriminator.

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For the practical calculations of the frequency of ejections the parameters, entering formula (3.6), are conveniently expressed through the spectral density of process x(t):

$$\nu=\frac{\omega_{II}}{2\pi}e^{-\gamma^2/2\sigma_x^2}, \qquad (3.7)$$

where ω_{II} - root-mean-square frequency of process x(t), determined by the expression

$$\omega_{II}^2=-\frac{r''(0)}{\sigma_x^2}=\frac{\int\limits_0^\infty \omega^2 N(\omega)\,d\omega}{\int\limits_0^\infty N(\omega)\,d\omega}; \qquad (3.8)$$

$N(\omega)$ - the spectral density of process x(t), connected with the correlation function $r(\tau)$ with Fourier transform (1.1); σ_x^2 - variance of error of the tracking:

$$\sigma_x^2=\frac{1}{2\pi}\int\limits_0^\infty N(\omega)\,d\omega. \qquad (3.9)$$

When correlation function $r(\tau)$ does not have the second derivative in zero (integral in the numerator (3.8) it diverges), process x(t) is undifferentiable. The frequency of the ejections of this process above the level γ, strictly speaking, is equal to infinity. This is explained by the fact that in immediate proximity of γ the process x(t) in view of brokenness manages an infinite number of times to cross this level. How ver, if we are not interested in the microstructure of process x(t), then the frequency of its ejections must remain final. Therefore in certain cases can prove to be useful the artificial reception/procedure of the calculation of the frequency of ejections [2], according to which

$$\nu = \frac{1}{\sigma_x} e^{-\gamma^2/2\sigma_x^2} \left\{ \frac{1}{8\pi^3} \int_0^\infty \left| j\omega \tilde{N}(j\omega) - [j\omega \tilde{N}(j\omega)]_{\omega\to\infty} \right|^2 d\omega \right\}^{1/2}, \quad (3.10)$$

where function $\tilde{N}(j\omega)$ is such that $N(\omega)=\tilde{N}(j\omega)\tilde{N}(-j\omega)=|\tilde{N}(j\omega)|^2$.

It must be noted that formula (3.10) is very approximate and in certain cases can lead to the erroneous results.

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Usually in the servo systems due to the action of regular

dynamic disturbance/perturbation λ(t) (see Fig. 1.2) the mathematical expectation of process x(t) is excellent from zero. If in this case $m_x(t)=m_x=\text{const}$, then for calculating the frequency of the ejections of this process above the level γ can be used formulas (3.6)-(3.7), in which should be value γ replaced the equivalent threshold

$$\gamma_0=\gamma-m_x. \qquad (3.11)$$

Formulas. Let us give the resultant expressions for the frequency of the ejections of following error x(t) above the level γ in certain frequently encountered systems.

Let the servo system have the block diagram, depicted in Fig. 1.2. If the characteristic of discriminator F(x) is linear with slope/transconductance S (see Fig. 3.1), then depending on the operational gear ratio/transmission factor of feedback loop K(p) we have:

1. First-order system with the ideal integrator [K(p)=K/p]:

$$\nu=\frac{KS}{2\pi}\exp\left[-\frac{2S\gamma^2}{KN_0}\right]. \qquad (3.12)$$

2. First-order system with the integrating filter [K(p)=K/(1+pT)]:

$$\nu=\frac{1+KS}{2\pi T}\exp\left[-\frac{2\gamma^2 T(1+KS)}{N_0K^2}\right]. \qquad (3.13)$$

3. System of the second order with the integrator and the proportional-integrating filter [K(p)=K(1+pT₁)/p(1+pT)]:

$$v=\frac{1}{2\pi}\sqrt{\frac{KS}{T}\frac{(1-n-k_0)^2+k_0}{1+k_0}}\exp\left[-\frac{2\gamma^2S(1+KSTn)}{KN_0(1+k_0)}\right], \quad (3.14)$$

$$n=T_1/T,\ k_0=KSTn^2.$$

In the particular case of integrating filter (n=0) we have

$$v=\frac{1}{2\pi}\sqrt{\frac{KS}{T}}\exp\left[-\frac{2\gamma^2S}{KN_0}\right]. \quad (3.15)$$

4. System with astaticism of second order [$K(p)=K(1+pT_1)/p^2$]:

$$v=\frac{1}{2\pi}\sqrt{KS\frac{(1+KST_1^2)^2}{1+(KST_1^2)^2}}\exp\left[-\frac{2\gamma^2S^2T_1}{N_0(1+KST_1^2)}\right]. \quad (3.16)$$

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5. System of third order with filter $K(p)=K/p(1+pT)(1+pT_1)$:

$$v=\frac{1}{2\pi}\sqrt{\frac{KS}{T+T_1}}\exp\left[-\frac{2\gamma^2S(T+T_1-KSTT_1)}{KN_0(T+T_1)}\right].$$

6. System of third order with filter $K(p)=K(1+pT_1)/p^2(1+pT)$:

$$v=\frac{1}{2\pi}\sqrt{KS\frac{1+nKST_1^2}{1+KST_1^2}}\exp\left[-\frac{2\gamma^2S^2(T_1-T)}{N_0(1+KST_1^2)}\right],$$

$$n=T_1/T. \quad (3.17)$$

Example. Let us determine the probability of disruption/separation in the servo system (see Fig. 1.2) with the proportional-integrating filter [$K(p)=K(1+pT_1)/p(1+pT)$] and with the characteristic of discriminator $F(x)=Sx$ in the limits of aperture $-\gamma_0<x<\gamma_0$. Let the input dynamic disturbance/perturbation take the form $\lambda(t)=\lambda_0+\lambda_1 t$; noise $\xi(t)$ - white with a spectral density of N_0, which does not

depend on disagreement/mismatch x. Let us assume that at the beginning of observation the transient processes in the system had time to be established/installed.

The frequency of the ejections of process x(t) above the level γ to the considered/examined system is determined by expression (3.14). So that during the calculation of the probability of disruption/separation it would be possible to use formula (3.4), it is necessary to preliminarily centralize process of x(t) and to determine equivalent thresholds γ_{10}, γ_{20}. Conservative value of dynamic error is equal to $m_x=\lambda_1/KS$. Hence according to (3.11) we obtain equivalent threshold values $\gamma_{10}=-\gamma_0-\frac{\lambda_1}{KS}$, $\gamma_{20}=\gamma_0-\frac{\lambda_1}{KS}$. As a result for the probability of disrupting/separating the tracking taking into account (3.4) and (3.14) we have

$$P(\tau)=\frac{\tau}{2\pi}\sqrt{KST\frac{(1-n-k_0)^2+k_0}{1+k_0}}\times$$
$$\times\left\{\exp\left[-\frac{2\left(\gamma_0+\frac{\lambda_1}{KS}\right)^2 S(1+KSTn)}{KN_0(1+k_0)}\right]+\right.$$
$$\left.+\exp\left[-\frac{2\left(\gamma_0-\frac{\lambda_1}{KS}\right)^2 S(1+KSTn)}{KN_0(1+k_0)}\right]\right\}, \qquad (3.18)$$

where $k_0=KSTn^2$, $n=T_1/T$, $\tau=t_n/T$ — dimensionless time.

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In the particular case when is absent dynamic error $\lambda_1=0$ and

n<<1, result noticeably is simplified:

$$P(\tau)=\frac{\tau\sqrt{KST}}{\pi}\frac{\sqrt{1+k_э^3}}{1+k_э}\exp\left[-\frac{2\gamma_0^2 S(1+KSTn)}{KN_0(1+k_э)}\right]. \quad (3.19)$$

According to the obtained relationships/ratios is constructed the series/row of the dependences of the probability of disruption/separation on the parameters of servo system (Fig. 3.2 and 3.3). Fig. 3.2 depicts the dependence of the probability of disruption/separation in the system of the second order with integrating filter (n=0) on the dimensionless parameter $Y=KN_0/S\gamma^2_0$, which characterizes the relation of the power of noise and signal at the output of discriminator. During the calculation it was accepted: KST=0.2, τ=1, λ_1=0. Let us recall that with KST<0.25 and n=0 the transient processes in the system carry aperiodic character. In Fig. 3.2 solid line constructed the approximate dependence, by dotted line - a more precise dependence, found by the simulation of servo system on the digital computer (TsVM [ЦВМ - digital computer]). The methodology of this simulation is presented into S 6.2. From the comparison of graphs is visible their asymptotic convergence with Y→0. This confirms the assumption made at first about the fact that on the sufficiently small noise level $\xi(t)$ the ejections of following error for the levels γ_1, γ_2 are subordinated to Poisson distribution.

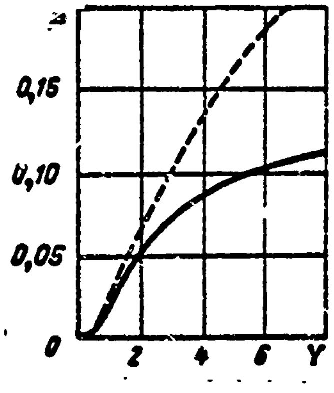

Fig. 3.2.

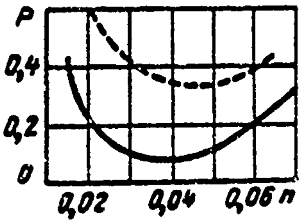

Fig. 3.3.

Fig. 3.2. Probability of disrupting/separating tracking in linear system.

Fig. 3.3. Probability of disrupting/separating tracking in linear system with proportional-integrating filter.

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In Fig. 3.3 is constructed the dependence of the probability of disrupting/separating the tracking in the system with the proportional-integrating filter on the relation of time constants $n=T_1/T$ in the following parameters of regulating circuit: KST=640, $Y=KN_0/S\gamma^2{}_0=6.4$, $\tau=1$, $\lambda_1=0$. With continuous line is constructed the curve, calculated by formulas (3.18), (3.19). Dotted line there constructed the analogous dependence, found by simulation on TsVM.

From the comparison of curves it is evident that for proportional-integrating filter ($n \neq 0$) of formula (3.18)-(3.19) they give a large error during the determination of the probability of disruption/separation, than for the integrating filter (n=0, Fig. 3.2). The relative disagreement between approximate value of the probability of disruption/separation, found from formulas (3.18)-(3.19), and precise value in the system with($n \neq 0$)does not vanish even with Y→0. This is explained by the fact that with($n \neq 0$) process x(t) is nondifferential; therefore it does not have the final frequency of ejections. The determination of the frequency of eje- ions from approximation formula (3.10) introduces appreciable error into the value of the probability of disruption/separation. Analogous result give formulas (3.12), (3.13) and (3.16). However, the given methodology it is expedient to use for the approximate calculations at the initial stage of the design of the systems of tracking, since with comparative simplicity of linings/calculations it gives qualitatively accurate picture and it makes it possible to determine acceptable noise level at the output of discriminator with an accuracy to 20-30% in the stress/voltage. Thus, in the example examined with n=0.04 an error in the determination of the noise voltage, which calls the probability of disruption/separation P=0.1 within the dimensionless time τ=1, was about 20%.

In the systems with the low coefficients of KST<10 is possible

even the qualitative disturbance/breakdown of the dependence of the probability of disruption/separation on the parameters of the proportional-integrating filter. For example, after calculating from formula (3.18) the system, which possesses KST=0.2 we will obtain that the probability of disruption/separation is reduced with an increase in relation n up to n=0.9. At the same time a more precise calculation and experimental check lead to the inverse dependence - the probability of disruption/separation noticeably increases with increase in n. Therefore with the low factors of amplification of the ring of servo system one ought not to use for the calculations of formula (3.18)-(3.19).

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Analysis of nonlinear systems. For calculating small probabilities of disrupting/separating the tracking in the nonlinear stationary regulating circuits remains valid formula (3.4), which escape/ensues from the Poisson distribution. The difficulty of the analysis of such systems consists in the calculation of the frequency of the ejections of process x(t) above the level γ that it is connected with the determination of two-dimensional probability density $w(x, \dot{x})$. With the dependence of spectral density $N_{\bullet}(x)$ of noise $\xi(t)$ from disagreement/mismatch x or in the systems with the nonlinear characteristics of discriminators F(x) the error

distribution of tracking w(x, t) differs from normal. In these cases it is not possible to directly use formulas (3.6), (3.[illegible] for calculating the frequency of ejections.

The frequency of disruptions/separations in the nonlinear systems under specific conditions sufficiently accurately can be calculated with the help of the methods of the theory of Markov processes. This approach to the determination of mean time to disruption/separation m_t, unambiguously connected with the frequency of disruptions/separations by the dependence

$$\nu_1 + \nu_2 = \frac{1}{m_t},$$

is examined into S 5.3. The complexity of the theory of Markov processes frequently makes it necessary to be converted to the simpler, although to the less precise receptions/procedures of analysis.

One of them can be the method of reference system with the subsequent use of correlation methods of analysis examined in this paragraph. In particular, for the linearization of the discriminatory characteristic F(x) can be used the method of statistical linearization [7], widely used during the research of nonlinear regulating circuits. To reduce the dependence of spectral density $N_e(x)$ on disagreement/mismatch x is possible, for example, by the replacement of the real disturbance/perturbation $\xi(t)$ with certain

equivalent $\xi_0(t)$ with the constant spectral density

$$N_{00}=\int_{\gamma_1}^{\gamma_2} N_0(x)\,w(x)\,dx, \tag{3.20}$$

where w(x) - the probability density of the error distribution of tracking, which in the first approximation, can be assumed/set by normal.

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In the systems of the first and second order with the integrating filter is feasible the following simple method of the linearization of the characteristic of discriminator. As shown in S 3.2, the probability of disrupting/separating the tracking in such systems very weakly depends on the form of discriminatory characteristic, and it is determined in essence by the areas, included between the point of stable equilibrium x_A and the boundaries of the aperture of discriminator. Thus, the initial nonlinear characteristic F(x) of discriminator can be substituted linear with slope/transconductance $S=F'(x_A)$ and by the boundaries of aperture, determined from the formulas:

with $f_0 \geq 0$

$$\gamma_{10}=-\sqrt{\frac{2}{S}\int_{x_A}^{\gamma_1}[F(x)-f_0]\,dx},$$

$$\gamma_{20}=\sqrt{\frac{2}{S}\int_{x_A}^{x_2}[F(x)-f_0]\,dx}, \tag{3.21}$$

with $f_0 \leq 0$

$$\gamma_{10} = -\sqrt{\frac{2}{S}\int_{x_A}^{x_1}[F(x)-f_0]\,dx},$$

$$\gamma_{20} = \sqrt{\frac{2}{S}\int_{x_A}^{\gamma_0}[F(x)-f_0]\,dx},$$

where $f_0=F(x_A)$ - constant detuning in the system, caused by the action of dynamic disturbance/perturbation $\lambda(t)$; x_A and x_1 -respectively the point of the stable and unstable equilibrium, determined with $N_0(x)$=const, from the equation

$$F(x)-f_0=0. \qquad (3.22)$$

The method of the linearization of characteristic F(x) examined is applicable also in the system of the second order with the proportional-integrating filter, if $KSTn \gtrsim 100$ or $KSTn^3 \ll 1$ [62], and in the system with astaticism of second order [67] with $KST^2{}_1 \geq 10$.

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Furthermore, in systems whose linear section of the discriminatory characteristic near the point of stable equilibrium exceeds $^1/_3$ of the aperture, the method of linearization examined leads to an error in the determination of signal-to-noise ratio from the power not more

than 20% with any relationships/ratios between the parameters of system [62, 67].

After the linearization of regulating circuit the probability of disrupting/separating the tracking in it is determined from formula (3.4).

Example. Let us consider the servo system of the second order with the integrating filter. The characteristic of the discriminator

$$F(x)=\begin{cases} S_0x & \text{(1) при } |x|<\gamma_0/2, \\ S_0(\gamma_0-x) & \text{(1) при } \gamma_0/2\leqslant x\leqslant\gamma_0, \\ -S_0(\gamma_0+x) & \text{(1) при } -\gamma_0/2\geqslant x\geqslant-\gamma_0, \\ 0 & \text{(1) при } |x|\geqslant\gamma_0, \end{cases} \quad (3.23)$$

Key: (1). with.

and spectral density N_0 does not depend on x. Let us determine the probability of disrupting/separating the tracking in the absence of dynamic error in system.

Following the methodology presented, let us replace the objective parameter of discriminator (3.23) with equivalent linear characteristic with slope/transconductance S_0, after assuming in accordance with (3.21) $\gamma_э=\gamma_0/\sqrt{2}$:

$$F_э(x)=\begin{cases} S_0x & \text{(1) при } |x|<\gamma_э, \\ 0 & \text{(1) при } |x|>\gamma_э. \end{cases}$$

Key: (1). with.

Formula for determining the probability of disruption/separation taking into account (3.4) and (3.15) in this case takes the form

$$P(t_{н})=\frac{t_{н}}{\pi T}\sqrt{KS_{0}T}\exp\left[-\frac{\gamma_{0}^{2}S_{0}}{KN_{0}}\right]. \qquad (3.24)$$

On Fig. 3.4 solid lines constructed those calculated with the help of (3.24) the dependence of the probability of disruption/separation P on the dimensionless coefficient of $Y=KN_0/\gamma^2{}_0S_0$ with different KS_0T.

The time of observation was proposed by such that $t_{н}/T=1$. More precise results obtained with the help of the simulation of initial nonlinear system on TsVM are shown in the figure by dotted line. From the comparison of curves it is evident that for the system of the second order with the integrating filter calculation of the probability of disruption/separation by the methods of correlation theory gives good results. This confirms assumption about the fact that in such systems is permitted the replacement of objective parameter F(x) of aperture linear with the equivalent change in accordance with (3.31). However, an error in the replacement increases with the decrease of the central linear section of characteristic F(x).

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If in the system is used the proportional-integrating or active integrating filter, then in the general case it is not possible to produce the replacement of the initial characteristic F(x) linear on the criterion of the equality of area under curve F(x). In this case it is necessary to use any other methods of linearization, for example on statistical criteria [7]. However, an error in these methods is comparatively great, which noticeably reduces the accuracy of the determination of the probability of disruption/separation by the methods of the theory of ejections. ' is it forces to be converted and to more precise methods of analysis. Most promising of them is the method of determining the probability of disruption/separation on the basis of the theory of Markov processes.

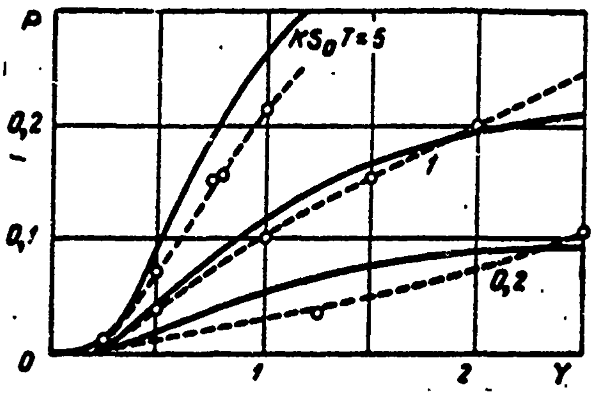

Fig. 3.4. Probability of disrupting/separating the tracking in the nonlinear system of the second order with the integrating filter.

3.2. Analysis of disruption/separation in the fixed systems with the help of the theory of Markov processes.

If following error x(t) is the component of the Markov process x(t), then the probability of disruption/separation is determined as a result of solving the boundary-value problem for the equation of Fokker-Planck (2.27) with the absorbing boundaries, situated on the edges of the aperture of discriminator.

In this paragraph is examined the method of approximate solution of the equation of Fokker-Planck whose basic ideas were for the first time proposed by Kramers into 1940 during the analysis of Brownian motion in field of force.

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Investigating the behavior of Brownian particles in the medium with the high viscosity and reducing the problem to the solution of the one-dimensional equation of Fokker-Planck, Kramers in work [27] found the solution of this equation taking into account the series/row of limitations to the form of field of force.

In 1943 Chandrasekar analyzed some particular cases of the behavior of the Brownian particles, described stochastic differential equation of second order [20].

Subsequently of the idea of the work of Kramers and Chandrasekhar it was possible to use for the analysis of the disruption/separation of tracking in the regulating circuits. This method was developed in the work of V. L. Lebedev, N. V. Belousovoy [55, 71] and S. V. Pervachev [62, 67]. However, it is not universal and at present it makes it possible to analyze the systems only of first and partially second order. Nevertheless method deserves attention, since with a comparatively small volume of calculations it makes it possible to obtain the series/row of practically important results.

Let us consider this method, gradually complicating the

structure of the device/equipment of automatic control.

1. Systems of first-order tracking.

Formulation of the problem. Let us determine the probability of disrupting/separating the tracking in the system (Fig. 3.5), described by differential first-order equation

$$\frac{dx}{dt}=\frac{d\lambda}{dt}-KF(x)+K\sqrt{N_0}\xi(t). \qquad (3.25)$$

Let the dynamic disturbance/perturbation $\lambda(t)$ be such, that $d\lambda/dt=\lambda_1=\text{const}$.

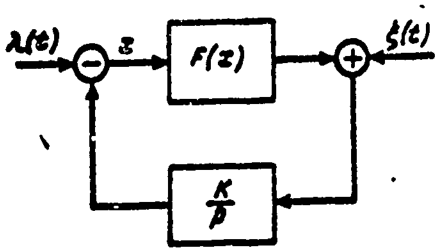

Fig. 3.5. Block diagram of servo system with one integrator.

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Let us register the equation of Fokker-Planck for the probability density of process x(t):

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial}{\partial x}\{K[F(x)-\Lambda]w\}+\frac{K^2N_0}{4}\frac{\partial^2 w}{\partial x^2}, \qquad (3.26)$$

where $\Lambda=\lambda_1/K$.

Understanding by the disruption/separation of tracking the first output of coordinate x beyond the limits γ_1, γ_2 the aperture of discriminatory characteristic, let us supplement equation (3.26) with the boundary conditions

$$w(\gamma_1, t)=w(\gamma_2, t)=0. \qquad (3.27)$$

Let us introduce into the examination the flow of probability density $\Pi(x, t)$ and, using divergent form (2.36) of the equation of Fokker-Planck, let us register

$$\Pi(x,t)=-K[F(x)-\Lambda]w(x,t)-\frac{K^2N_0}{4}\frac{\partial w(x,t)}{\partial x}. \qquad (3.28)$$

Let us assume that the transient processes in the system up to the moment/torque of the beginning of observation will be finished or the time of their establishment will compose the insignificant part of entire time of observation t_H. Then, if the probability of disruption/separation is sufficiently small $(P(t_n) \leq 0{,}2)$, probability density w(x, t) little is changed for the time of observation; therefore

$$\Pi(x, t) \approx \Pi = \mathrm{const}. \tag{3.29}$$

Relationship/ratio of Kramers. Let us introduce function $\mathcal{P}(x)$ (Fig. 3.6), such, that

$$K[F(x)-A] = \frac{d\mathcal{P}(x)}{dx}. \tag{3.30}$$

Function $\mathcal{P}(x)$ is called potential or potential function. Actually/really, if we consider w(x, t) as the density of distribution of Brownian particles along coordinate x, then value K [F(x)-A] characterizes the regular force, which functions on the particles, and $\mathcal{P}(x)$ - potential field in which are located the particles. After expressing the flow through the potential, on the basis (3.28) and (3.30) we will obtain

$$\Pi = -w(x)\frac{d\mathcal{P}(x)}{dx} - \frac{K^2N_0}{4}\frac{dw(x)}{dx}. \tag{3.31}$$

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Latter/last equality can be represented in the form

$$\Pi e^{4\mathcal{P}(x)/K^2N_0} = -\frac{K^2N_0}{4}\frac{d}{dx}\left[w(x)e^{4\mathcal{P}(x)/K^2N_0}\right], \tag{3.32}$$

in what not difficult to be convinced, aperture its right side.

Integrating both parts of expression (3.32) on x in the arbitrary limits from x_A to x_B, we will obtain the relationship/ratio of Kramers [27]

$$\Pi = \frac{K^3 N_0 \left[w(x) e^{4\mathcal{P}(x)/K^3 N_0} \right]_{x_B}^{x_A}}{4 \int\limits_{x_A}^{x_B} e^{4\mathcal{P}(x)/K^3 N_0} dx}, \qquad (3.33)$$

playing important role in the solution of boundary-value problems for the equation of Fokker-Planck.

Determination of the probability of disruption/separation. Let us assume that x_A is the point of stable equilibrium in the system (see Fig. 3.6), and point x_B coincides with one of the absorbing boundaries $x_B = \gamma_2$. Let us introduce potential $\mathcal{P}(x)$ by such form, in order to $\mathcal{P}(x_A) = 0$. This always can be done, since addition to function $\mathcal{P}(x)$ of constant value will not influence density distribution of probability w(x,t).

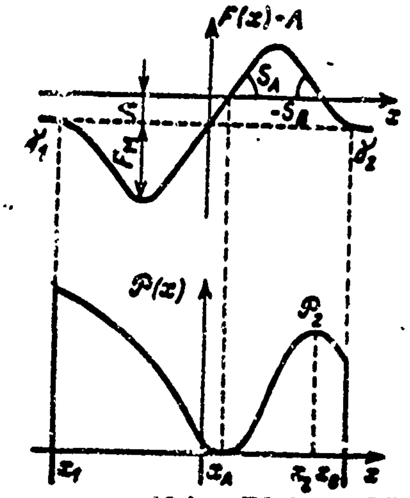

Fig. 3.6. Coefficient of removal/drift and the potential function of servo system.

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Let us note also that in view of boundary condition (3.27) $w(x_B)=0$. This makes it possible to register relationship/ratio (3.33) in the following form:

$$\Pi=\frac{K^2N_0w(x_A)}{4\int\limits_{x_A}^{x_B} e^{4\mathcal{P}(x)/K^2N_0}dx}. \qquad (3.34)$$

Usually the point of stable equilibrium is arranged/located in the linear section of the characteristic of discriminator. Therefore approximately it is possible to consider that near x_A the density of probability obeys the normal distribution law

$$w(x)|_{x\sim x_A} \approx \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(x-x_A)^2}{2\sigma_x^2}\right],$$

where conservative value of dispersion σ_x^2 is determined by the relationship/ratio

$$\sigma_x^2 = \frac{N_0}{2\pi}\int_0^\infty \left|\frac{K(j\omega)}{1+S_A K(j\omega)}\right|^2 d\omega.$$

valid for the linear regulating circuits. With this K(jω) - the complex gear ratio/transmission factor of the feedback loop of the servo system (see Fig. 1.2), S_A - mutual conductance of discriminator in region $x\sim x_A$. Taking into account that for the system K(jω)=K/jω in question, we obtain $\sigma_x^2 = KN_0/4S_A$. Thus,

$$w(x_A) \approx \sqrt{\frac{2S_A}{\pi KN_0}}. \qquad (3.35)$$

With the constant flow Π the probability of reaching/achievement by coordinate x for time t_x of point γ_2 is determined from the formula

$$P_2 = \Pi t_x. \qquad (3.36)$$

Hence taking into account (3.34) and (3.35)

$$P_2(t_x) \approx Kt_x \sqrt{S_A KN_0}\left[2\sqrt{2\pi}\int_{x_A}^{\gamma_2} e^{4\mathcal{F}(x)/KN_0}\,dx\right]^{-1}.$$

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Producing analogous conversions for the boundary γ_1 and taking into account that with small probabilities of disruption/separation the ejections of process x(t) beyond the boundaries γ_1 and γ_2 - event independent, let us register the resultant expression for the probability of disruption/separation for time $t_н$:

$$P(t_н) \approx \frac{K t_н \xi \sqrt{S_A K N_0}}{2\sqrt{2\pi}} \left\{ \left[\int_{x_A}^{\gamma_2} e^{4\mathcal{F}(x)/K^2 N_0} dx \right]^{-1} - \left[\int_{x_A}^{\gamma_1} e^{4\mathcal{F}(x)/K^2 N_0} dx \right]^{-1} \right\}. \tag{3.37}$$

The obtained relationship/ratio is correct with any form of discriminatory characteristic. It is important only so that in the vicinity of the point of stable equilibrium x_A characteristic F(x) would be close to the linear.

The integrals, entering expression (3.37) if necessary can be accurately calculated by the analytical or graphic method. Let us isolate the case, which is frequently encountered in the practice when calculation according to formula (3.37) substantially is simplified.

Let us assume that the characteristic of discriminator is the odd function F(-x)=-F(x). let us take for the definition, that the input dynamic disturbance/perturbation λ(t) causes positive detuning

$\Lambda>0$.

The integrals, entering in (3.37), virtually are determined by the small regions x near the maximums of potential field $\mathcal{P}(x)$ (see Fig. 3.6), since with small probabilities of disruption/separation $4\mathcal{P}(x)/K^2N_0\gg 1$. Let us expand function $\mathcal{P}(x)$ in the vicinity of its maximums in the Taylor series. Characteristic F(x) near these points in many practical cases can be approximated by linear section with slope/transconductance $S_B=-dF/dx$, therefore let us take into account only two terms of the expansion

$$\mathcal{P}(x)\approx\begin{cases}\mathcal{P}_1-\frac{1}{2}KS_B(x-x_1)^2 \overset{(1)}{\text{при}}\ x\sim x_1,\\ \mathcal{P}_2-\frac{1}{2}KS_B(x-x_2)^2 \overset{(1)}{\text{при}}\ x\sim x_2,\end{cases}\qquad (3.38)$$

Key: (1). with.

where $\mathcal{P}_1=\mathcal{P}(x_1)$ and $\mathcal{P}_2=\mathcal{P}(x_2)$ - value of the potential thresholds.

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With the low value the detuning $\Lambda \sim 0$ position x_2 of the maximum of potential field virtually coincides with the boundary γ_2.

In this case, taking into account (3.38), we obtain

$$-\int_{x_A}^{\gamma_1} e^{4\mathcal{P}(x)/K^2N_0}\,dx = \int_{x_A}^{\gamma_2} e^{4\mathcal{P}(x)/K^2N_0}\,dx \approx$$
$$\approx \frac{1}{2}\int_{-\infty}^{\infty} \exp\left\{\frac{4}{K^2N_0}\left[\mathcal{P}_m - \frac{1}{2}KS_B(x-\gamma)^2\right]\right\}dx =$$
$$= \frac{1}{2}\sqrt{\frac{\pi KN_0}{2S_B}}\,e^{4\mathcal{P}_m/K^2N_0},$$
$$\gamma = \gamma_2 = -\gamma_1,$$

and probability of disruption/separation is equal to

$$P(t_n) \approx \frac{2K\sqrt{S_A S_B}\,t_n}{\pi}\,e^{-4\mathcal{P}_m/K^2N_0}, \tag{3.39}$$

where

$$\mathcal{P}_m = K\int_0^{\gamma} F(x)\,dx.$$

In other limiting case with the large detuning $\Lambda \sim \gamma$ parabolic approximation (3.38) of potential field near point x_2 can be continued into the region infinite x. Then

$$-\int_{x_A}^{\gamma_1} e^{4\mathcal{P}(x)/K^2N_0}\,dx \approx \frac{1}{2}\sqrt{\frac{\pi KN_0}{2S_B}}\,e^{4\mathcal{P}_1/K^2N_0}, \tag{3.40}$$

$$\int_{x_A}^{\gamma_2} e^{4\mathcal{P}(x)/K^2N_0}\,dx \approx \sqrt{\frac{\pi KN_0}{2S_B}}\,e^{4\mathcal{P}_2/K^2N_0}, \tag{3.41}$$

as a result the probability of disruption/separation is equal to

$$P(t_n) \approx \frac{K\sqrt{S_A S_B}t_n}{2\pi}\left\{2e^{-4\mathcal{P}_1/K^2N_0} + e^{-4\mathcal{P}_2/K^2N_0}\right\}, \quad (3.42)$$

where

$$\mathcal{P}_2 = K\int_{x_A}^{\gamma_2}[F(x) - \Lambda]\,dx;\ \mathcal{P}_1 = K\int_{x_A}^{x_0}[F(x) - \Lambda]\,dx.$$

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Frequently during the analysis of the disruption/separation of tracking the potential threshold near the point γ_1 they approximate not by one branch of parabola as is done in (3.40), but two, extending integration limits on $\pm\infty$. This gives certain further error in the determination of the probability of disruption/separation. It is possible to disregard it if Λ is sufficiently great so that one of the potential thresholds would be noticeably higher than another. Then instead of (3.42) and (3.39) we obtain one overall dependence

$$P(t_n) \approx \frac{K\sqrt{S_A S_B}t_n}{2\pi}\left[e^{-4\mathcal{P}_1/K^2N_0} + e^{-4\mathcal{P}_2/K^2N_0}\right]. \quad (3.43)$$

Taking into account that a decisive effect on value $P(t_n)$ in formulas (3.39), (3.42), (3.43) have the exponential terms, it is

possible to do the conclusion that in first-order systems the probability of disruption/separation in essence depends not on the form of the discriminatory characteristic F(x), but on the height of potential barriers $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e., on the areas under the branches of discriminatory characteristic. This can be used for replacing the objective parameter F(x) of linear when the heights/altitudes of the potential thresholds in the linearized system will remain equal to barrier heights in the reference system. The latter is achieved by the introduction of equivalent boundaries γ_{1o} and γ_{2o} according to formulas (3.21). The linearization of system makes it possible to use for the proximate analysis of the disruption/separation of tracking methods of the theory of ejections.

Account of the fluctuating characteristic of discriminator. Let us spread the method of determining the probability of disruption/separation presented to the case when the spectral density of white noise at the output of discriminator depends on disagreement/mismatch x. Let the differential equation, which describes the behavior of system, take the form

$$\frac{dx}{dt}=a(x)+b(x)\xi^{0}(t),$$

where in contrast to (3.25) the intensity of white noise is the function of following error x.

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For this case let us compose the equation of Fokker-Planck, using a form of R. L. Stratonovich's recording:

$$\frac{\partial w(x,t)}{\partial t}=-\frac{\partial}{\partial x}\left[\left(a+\frac{1}{4}\frac{dB}{dx}\right)w\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}[Bw], \quad (3.44)$$

where

$$B(x)=\frac{b^2(x)}{2}.$$

Expression for the flow of probability density takes the form

$$\Pi=a(x)w(x,t)-\frac{1}{2}B(x)\frac{dw}{dx}-\frac{1}{4}w(x,t)\frac{dB}{dx}. \quad (3.45)$$

Let us find the solution of equation (3.44), which little varies in the time. As shown in [14], the steady-state solution of equation (3.44) takes the form

$$w(x)=\frac{C}{\sqrt{B(x)}}\exp\left[2\int^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right], \quad (3.46)$$

where C - constant, determined from standardization condition.

On the basis (3.45) and (3.46) it is possible to obtain the expression for the stationary flow, which, as it follows from [55], takes the form

$$\Pi=-\sqrt{B(x)}\frac{1}{2}\exp\left[2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right]\frac{d}{dx}\left\{\sqrt{B(x)}\,w(x)\times\right.$$

$$\left.\times\exp\left[-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right]\right\}. \quad (3.47)$$

Conformity (3.45) and (3.47) is not difficult to check by direct differentiation. From (3.47) it follows that

$$\frac{2\Pi}{\sqrt{B(x)}}\exp\left[-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right]=$$

$$=-\frac{d}{dx}\left\{\sqrt{B(x)}\,w(x)\exp\left[-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right]\right\}. \quad (3.48)$$

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For det ing the probability of overcoming by trajectory x(t) of the potent threshold at point x_2 (see Fig. 3.6), let us integrate both parts of expression (3.48) with respect to x in the limits from x_A (point of stable equilibrium) to absorbing boundary $x_B=\gamma_2$. As a result we will obtain the following expression for the flow of probability density:

$$\Pi=\frac{\left\{\sqrt{B(x)}\,w(x)\exp\left[-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right]\right\}_{x_B}^{x_A}}{2\cdot\int_{x_A}^{x_B}\frac{1}{\sqrt{B(x)}}\exp\left\{-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}d\zeta\right\}dx}, \quad (3.49)$$

which is the generalization of the relationship/ratio of Kramers (3.33).

Taking into account that $w(x_B)=0$, we have

$$\Pi = \frac{\sqrt{B(x_A)}\, w(x_A)}{2\int\limits_{x_A}^{l_0} \frac{1}{\sqrt{B(x)}} \exp\left[-2\int\limits_{x_A}^{x} \frac{a(\zeta)}{B(\zeta)} d\zeta\right] dx}. \qquad (3.50)$$

Relationships/ratios (3.50) and (3.36) make it possible to determine the probability of achieving the absorbing boundary γ_2. However, direct calculations according to formula (3.50) are bulky; therefore let us produce further simplification in this expression.

Density distribution of probability near the point of stable equilibrium usually differs little from the normal

$$w(x)\big|_{x \sim x_A} = \frac{C}{\sqrt{B(x)}} \exp\left[2\int\limits_{x_A}^{x} \frac{a(\zeta)}{B(\zeta)} d\zeta\right] \approx$$

$$\approx \frac{1}{\sqrt{2\pi}\,\sigma_A} \exp\left[-\frac{(x-x_A)^2}{2\sigma_A^2}\right],$$

whence

$$w(x_A) = \frac{1}{\sqrt{2\pi}\,\sigma_A} = \sqrt{\frac{B'(x_A) - 4a'(x_A)}{4\pi B(x_A)}}. \qquad (3.51)$$

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Further reasonings differ little from case of B(x) =const. Considering internal integral in expression (3.50) as certain potential field, which has maximums at points x_1 and x_2, let us introduce the approximation

$$\mathcal{F}(x)=-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}\,d\zeta=\begin{cases}\mathcal{F}_1-\frac{1}{2}S_1(x-x_1)^2 & \text{при}^{(1)}\ x\sim x_1,\\ \mathcal{F}_2-\frac{1}{2}S_2(x-x_2)^2 & \text{при}^{(1)}\ x\sim x_2,\end{cases}$$

where

$$\mathcal{F}_i=\mathcal{F}(x_i),\ S_i=-\frac{d}{dx}\left[\frac{2a(x)}{B(x)}\right]_{x=x_i},\ i=1,2.$$

Key: (1). with.

The value of external integral in (3.50) virtually is determined by the behavior of function $\mathcal{F}_1(x)$ in the small region about the maximum of potential field; therefore

$$\int_{x_A}^{x_B}\frac{1}{\sqrt{B(x)}}\exp\left[-2\int_{x_A}^{x}\frac{a(\zeta)}{B(\zeta)}\,d\zeta\right]dx\approx\frac{1}{\sqrt{B(x_2)}}\times$$

$$\times e^{\mathcal{F}_2}\int_{-\infty}^{x_B}\exp\left[-\frac{1}{2}S_2(x-x_2)^2\right]dx. \qquad (3.52)$$

With small detuning $\Lambda\sim0$ the point of the unstable equilibrium x_2 is close to absorbing boundary $x_B\approx\gamma_2$ therefore upper integration limit in (3.52) can be replaced by x_2.

With large detuning $\Lambda\sim F_M/2$ upper integration limit can be approximately increased to infinity. As a result we will obtain following expression for the flow of probability density through the right boundary:

$$\Pi\approx s\sqrt{B(x_A)}\,w(x_A)\sqrt{\frac{S_2B(x_2)}{2\pi}}\,e^{-\mathcal{F}_2},$$

where

$$\varepsilon = \begin{cases} 1 & \text{при малых расстройках,}^{(1)} \\ 1/2 & \text{при больших расстройках.}^{(2)} \end{cases}$$

Key: (1). with small maladjustments

(2). with large maladjustments

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The flow through the left boundary the value of detuning is independent of determined by the relationship/ratio

$$\Pi \approx -\sqrt{B(x_A)}\, w(x_A) \sqrt{\frac{S_1 B(\gamma_1)}{2\pi}}\, e^{-\mathcal{P}_1},$$

Hence on the basis (3.36) and (3.51) is obtained the resultant expression for the probability of disruption/separation, for time t_n:

$$P(t_n) \approx \frac{t_n}{2\pi} \sqrt{\frac{1}{2} B''(x_A) - 2a'(x_A)}\, [\sqrt{B(\gamma_1) S_1}\, e^{-\mathcal{P}_1} + \\ + \varepsilon \sqrt{S_2 B(x_2)}\, e^{-\mathcal{P}_2}]. \qquad (3.53)$$

Points x_A and x_2 are respectively the points of the stable and unstable equilibrium of system. They can be determined from the condition

$$a(x) = -\frac{1}{4}\frac{dB(x)}{dx}.$$

Conclusions/outputs. As a result of the analysis conducted are obtained expressions (3.39), (3.42), (3.43), (3.53), the making it possible to approximately determine the probability of disruption/separation trackings in the nonlinear first-order systems.

The basic assumptions, done in the analysis run, are reduced to the following. It is assumed that the probability of disrupting/separating the tracking in the system is sufficiently small $(P(t_H) < 0{,}1 \div 0{,}2)$, therefore the error distribution of tracking w(x) little varies for the time of observation t_H. The characteristic of discriminator F(x) has linear section near the point of stable equilibrium x_A, which is used during the calculation of probability density at this point $w(x_A)$.

From the obtained relationships/ratios it follows that the probability of disrupting/separating the tracking in first-order systems in essence is determined by the exponential factors whose indices depend on the height/altitude of potential thresholds $\mathcal{P}_1$ and $\mathcal{P}_2$.

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Since values $\mathcal{P}_1$ and $\mathcal{P}_2$ are determined only by the area, included under the discriminatory characteristic, then it is possible to consider that the probability of disrupting/separating the tracking in first-order systems virtually does not depend on the form of characteristic F(x) with those fixed/recorded $\mathcal{P}_1$ and $\mathcal{P}_2$.

2. System of the second order with the integrating filter.

Formulation of the problem. Let us consider servo system with the gear ratio/transmission factor of the circuit of feedback

$$K(p)=\frac{K}{p_0(1+pT)}.$$

Stochastic differential equation for the following error x(t) of this system takes the form

$$T\frac{d^2x}{dt^2}+\frac{dx}{dt}+KF(x)=T\frac{d^2\lambda}{dt^2}+\frac{d\lambda}{dt}-K\sqrt{N_0}\xi(t), \quad (3.54)$$

where all designations are analogous to designations in (3.25).

After assuming dx/dt=y and $d\lambda/dt=\lambda_1$=const, let us compose on the basis of (3.54) the equation of Fokker-Planck for the two-dimensional probability density w(x, y, t):

$$\frac{\partial w}{\partial t}=\frac{1}{T}\frac{\partial}{\partial y}[yw]+\frac{K}{T}[F(x)-\Lambda]\frac{\partial w}{\partial y}-y\frac{\partial w}{\partial x}+\frac{K^2N_0}{4T^2}\frac{\partial^2 w}{\partial y^2}, \quad (3.55)$$

where

$$\Lambda=\frac{\lambda_1}{K}.$$

It is necessary to determine probability that for time t_m which passed from the moment/torque of inclusion/connection, trajectory x(t) at least one time fall outside the limits γ_1, γ_2 the aperture of discriminator.

Physical analogy. Let us consider the potential field

$$\mathcal{P}(x)=\frac{K}{T}\int_{x_A}^{x}[F(\zeta)-\Lambda]\,d\zeta, \quad (3.56)$$

where x_A - point of stable equilibrium in the system.

In contrast to first-order system equation (3.54) describes the behavior of the Brownian particles, which have finite mass T. Therefore particles, being located in the potential field, possess the specific inertia and cannot for the short time substantially change their trajectory.

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After achieving the maximum of potential threshold $\mathcal{P}_m$ (Fig. 3.7) and having positive speed, particles with the probability, close to one, are rolled up beyond the limits of the barrier (they surmount it). Thus, the disruption/separation of tracking can be identified not with the reaching/achievement by coordinate x(t) of boundaries γ_1, γ_2, but with output x(t) beyond the limits of the potential thresholds. Let us determine the probability of overcoming by process x(t) of the barrier, arranged/located at point x_B (Fig. 3.7).

Quasi-stationary solution. The solution of equation (3.55), found on the assumption that $\partial w/\partial t=0$, takes form [14, 20]

$$w(x, y) = C \exp\left[-\frac{2Ty^2}{K^2N_0} - \frac{4T}{K^2N_0}\mathcal{P}(x)\right], \quad (3.57)$$

where C—the constant, determined from standardization condition; $\mathcal{P}(x)$ - potential field, introduced by relationship/ratio (3.56).

Let us assume that near the point of stable equilibrium x_A the characteristic of discriminator is linear with slope/transconductance S_A and root-mean-square following error is small in comparison with the extent of linear section. With small probabilities of disruption/separation ($P(t_n) \lesssim 0{,}2$) two-dimensional density of distribution w(x, y, t) in region $x \sim x_A$ is approximately determined by expression (3.57), which taking into account linearity F(x) near x_A takes the form

$$w(x, y)\big|_{x \sim x_A} \approx \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{y^2}{2\sigma_y^2} - \frac{(x - x_A)^2}{2\sigma_x^2}\right], \quad (3.58)$$

where $\sigma_x^2 = KN_0/4S_A$, $\sigma_y^2 = K^2N_0/4T$ - dispersion of processes of x(t) and y(t).

Near the potential threshold the true distribution w(x, y) does not correspond to (3.58).

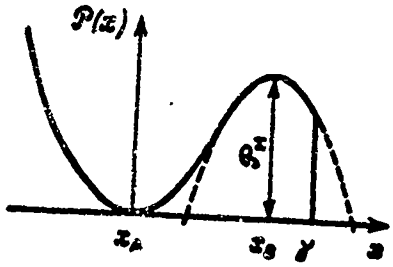

Fig. 3.7. Approximation of potential near the barrier.

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In [20] it is proposed to seek the quasi-stationary solution of equation (3.55) in the form

$$w(x, y)=\frac{Q(x, y)}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{y^2}{2\sigma_y^2}-\frac{4T}{K^2N_0}\mathcal{P}(x)\right], \quad (3.59)$$

where function Q(x, y) must satisfy the conditions

$$Q(x, y)\approx\begin{cases}1 \text{ при}^{(1)}\ x\sim x_A,\\ 0 \text{ при}^{(1)}\ x> x_B.\end{cases} \quad (3.60)$$

Key: (1). with.

Let us approximate potential field near x_B by the parabola (see Fig. 3.7):

$$\mathcal{P}(x)\approx\mathcal{P}_M-\frac{1}{2}\frac{S_BK}{T}(x-x_B)^2, \quad (3.61)$$

where $\mathcal{P}_M$ - height/altitude of the potential threshold; $S_B = -\left.\frac{dF(x)}{dx}\right|_{x=x_B}$ — mutual conductance of discriminator near x_B.

Taking into account (3.61) solution (3.59) near point x_B is determined by the expression

$$w(x, y)|_{x\sim x_B} = CQ(x, y)\exp\left\{-\frac{2Ty^2}{K^2N_0} + \frac{2S_B(x-x_B)^2}{KN_0}\right\}, \quad (3.62)$$

where

$$C = \frac{2}{\pi KN_0}\sqrt{\frac{TS_A}{K}}\, e^{-4T\mathcal{P}_M/K^2N_0}.$$

Let us introduce new variable/alternating $X = x - x_B$. In this case the steady-state equation of Fokker-Planck in vicinity $x\sim x_B$ will take the form

$$y\frac{\partial w}{\partial X} + \frac{KS_BX}{T}\frac{\partial w}{\partial y} = \frac{1}{T}\frac{\partial}{\partial y}(yw) + \frac{K^2N_0}{4T^2}\frac{\partial^2 w}{\partial y^2}. \quad (3.63)$$

and solution (3.62) is equal

$$w(x, y)|_{x\sim x_B} = CQ(X, y)\exp\left\{-\frac{2Ty^2}{K^2N_0} + \frac{2S_BX^2}{KN_0}\right\}. \quad (3.64)$$

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Substituting (3.64) in (3.63), we obtain equation for determining the unknown function Q(X, y):

$$y\frac{\partial Q}{\partial X} + \frac{KS_B}{T}X\frac{\partial Q}{\partial y} + \frac{y}{T}\frac{\partial Q}{\partial y} = \frac{K^2N_0}{4T^2}\frac{\partial^2 Q}{\partial y^2}. \quad (3.65)$$

The obvious solution of equation $Q(X, y)\equiv 1$ does not interest us, since it does not satisfy conditions (3.60), which for new variable X take the form

$$Q(X, y) \approx \begin{cases} 1 \text{ при}^{(1)}\ X \approx -(x_B - x_A), \\ 0 \text{ при } X\to\infty. \end{cases} \quad (3.66)$$

Key: (1). with.

Let us assume that the solution of equation (3.65), which satisfies boundary conditions (3.66), can be found in the form

$$Q(X, y) = Q(y - aX) = Q(z), \qquad (3.67)$$

where a - certain constant value. Substituting (3.67) in equation (3.65) and passing to the differentiation with respect to new to the variable/alternating z, we will obtain

$$-[(aT - 1)y - KS_B X]\frac{dQ}{dz} = \frac{K^2 N_0}{4T}\frac{d^2Q}{dz^2}. \qquad (3.68)$$

So that expression (3.68) would not contradict (3.67), necessary to assume

$$\frac{KS_B}{aT - 1} = a, \qquad (3.69)$$

as a result of what equation (3.68) takes the form

$$-(aT - 1)z\frac{dQ}{dz} = \frac{K^2 N_0}{4T}\frac{d^2Q}{dz^2}.$$

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Its solution is located by the direct integration

$$Q(z) = C_0 \int_{z_0}^{z} \exp\left[-\frac{2T(aT - 1)\zeta^2}{K^2 N_0}\right] d\zeta, \qquad (3.70)$$

where C_0, z_0 - constants, determined from conditions (3.66). In this case these conditions with the small error can be substituted by the following:

$$Q(X, y) \approx \begin{cases} 1 \ \text{при}^{(1)} \ X \to -\infty, \\ 0 \ \text{при}^{(2)} \ X \to \infty. \end{cases} \qquad (3.71)$$

Key: (1). with.

Hence finally we obtain

$$Q(z) = \sqrt{\frac{2T(aT-1)}{\pi K^2 N_0}} \int_{-\infty}^{z} \exp\left\{-\frac{2T(aT-1)\zeta^2}{K^2 N_0}\right\} d\zeta,$$

where $a = \frac{1}{2T} + \sqrt{\frac{1}{4T^2} + \frac{KS_B}{T}}$ — positive root of equation (3.69) [negative root does not satisfy conditions (3.71)].

Thus, the steady-state solution of the equation of Fokker-Planck near the potential threshold takes the form

$$w(X, y) \approx \frac{2\sqrt{2}T}{\pi K^2 N_0} \sqrt{\frac{S_A (aT-1)}{\pi K N_0}} \exp\left[-\frac{4T\mathcal{P}_M}{K^2 N_0}\right] \times$$
$$\times \exp\left[-\frac{2Ty^2}{K^2 N_0} + \frac{2S_B X^2}{K N_0}\right] \int_{-\infty}^{y-aX} \exp\left[-\frac{2T(aT-1)\zeta^2}{K^2 N_0}\right] d\zeta. \quad (3.72)$$

Probability of disruption/separation. The flow of probability density through point x_B is determined by the expression

$$\Pi(x_B) = \int_{-\infty}^{\infty} y w(X=0, y)\, dy. \quad (3.73)$$

After substituting (3.72) and (3.73), after integration we will in parts obtain expression for the flow through the potential threshold

$$\Pi(x_B) = \frac{1}{2\pi} \sqrt{\frac{S_A}{S_B}} \left[\sqrt{\frac{1}{4T^2} + \frac{KS_B}{T}} - \frac{1}{2T}\right] \times$$
$$\times \exp\left(-\frac{4T\mathcal{P}_M}{K^2 N_0}\right).$$

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Let us define the probability of disruption/separation in the presence of two potential thresholds as

$$P(t_{\text{н}}) = |\Pi_1| t_{\text{н}} + |\Pi_2| t_{\text{н}}, \qquad (3.74)$$

and let us register final formula for the probability of disrupting/separating the tracking

$$P(t_{\text{н}}) = \frac{t_{\text{н}}}{2\pi}\left\{\sqrt{\frac{S_0}{S_1}}\left[\sqrt{\frac{1}{4T^2} + \frac{KS_1}{T}} - \frac{1}{2T}\right]\exp\left(-\frac{4T\mathcal{P}_1}{K^2N_0}\right) + \right.$$
$$\left. + \sqrt{\frac{S_0}{S_2}}\left[\sqrt{\frac{1}{4T^2} + \frac{KS_2}{T}} - \frac{1}{2T}\right]\exp\left(-\frac{4T\mathcal{P}_2}{K^2N_0}\right)\right\}, \qquad (3.75)$$

where $t_{\text{н}}$ — time of observation; S_0 - mutual conductance of discriminator at the point of stable equilibrium of system; $S_{1(2)}$ - absolute values of the slope/transconductance of discriminatory characteristic in the vicinities of the potential thresholds; $\mathcal{P}_1$, $\mathcal{P}_2$ — heights/altitudes of the potential thresholds, determined by the relationships/ratios

$$\mathcal{P}_1 = \frac{K}{T}\int_{x_0}^{x_1}[F(x) - \Lambda]\,dx, \quad \mathcal{P}_2 = \frac{K}{T}\int_{x_0}^{x_2}[F(x) - \Lambda]\,dx;$$

x_1, x_2 - coordinates of the potential thresholds which depending on the sign of unbalance Λ coincide either with the point of unstable equilibrium $x_{\text{н}}$ or with the boundary of the aperture of the discriminatory characteristic

$$x_1 = \max(x_{\text{н}}, \gamma_1),\ x_2 = \min(x_{\text{н}}, \gamma_2).$$

Here for the certainty it is reported $\gamma_1<0$, $\gamma_2>0$. If, for example,

$\Lambda>0$, then $x_1=\gamma_1$, $x_2=x_H$. Coordinate x_H is determined from the conditions

$$F(x_H)-\Lambda=0,\ \left.\frac{dF(x)}{dx}\right|_{x=x_H}<0.$$

If is fulfilled inequality $\sqrt{KS_0T}\gg 1$, then formula (3.75) is simplified and takes the form

$$P(t_H)=\frac{t_H}{2\pi}\sqrt{\frac{KS_0}{T}}\left[\exp\left(-\frac{4T\mathcal{P}_1}{K^2N_0}\right)+\exp\left(-\frac{4T\mathcal{P}_2}{K^2N_0}\right)\right]. \quad (3.76)$$

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After comparing the obtained expression with (3.15), let us note that the probability of disruption/separation in the system in question is equal to the probability of disruption/separation in the linear system, if the heights/altitudes of the potential thresholds in both systems are identical. This can be achieved/reached, if we as the boundaries of the aperture of linear system take values $\gamma_{1э}$ and $\gamma_{2э}$ determined by relationships/ratios (3.21).

Account of the fluctuating characteristic of discriminator. The methodology of the determination of the probability of disrupting/separating the tracking presented taking into account the series/row of further limitations can be spread also to the case when the spectral density of effect $\xi(t)$ depends on disagreement/mismatch x [62].

Let in stochastic equation (3.54) $N_0=N_0(x)$ and $d\lambda/dt=\lambda_1=$const.

If the time constant T of the integrating filter is low, then equation (3.54) is degenerated into the first-order equation which was analyzed in the previous section. The probability of disruption/separation in this case is determined by dependence (3.53).

Let us consider another limiting case when T is great $(KS_AT\gg1)$.

As was already said in this paragraph, equation (3.54) describes the behavior of inertia Brownian particle in field of force KF(x). In this case the coefficient with dx/dt plays the role of friction. As can be seen from (3.54), with the high value T the role of friction is reduced. For the analysis of systems with small friction let us introduce into the examination the variable/alternating E, which characterizes energy of particle with the single mass in potential field $\mathcal{P}(x)$ [14]:

$$E=\frac{\dot{x}^2}{2}+\mathcal{P}(x), \tag{3.77}$$

where $\mathcal{P}(x)=\frac{K}{T}\int\limits_{x_A}^{x}[F(\zeta)-\Lambda]d\zeta;$ x_A — point of stable equilibrium of system.

Equation (3.54) taking into account (3.77) can be represented in the form of the following system:

$$\left.\begin{aligned}\frac{dx}{dt}&=\sqrt{2[E-\mathcal{P}(x)]},\\ \frac{dE}{dt}&=-\frac{2}{T}[E-\mathcal{P}(x)]+\sqrt{2[E-\mathcal{P}(x)]g(x)}\xi^{\circ}(t)\end{aligned}\right\} \quad (3.78)$$

where

$$g(x)=\frac{K^2}{T^2}N_0(x).$$

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Using the methodology, presented in S 2.3, let us compose the equation of Fokker-Planck for the two-dimensional probability density w(x, E):

$$\frac{\partial w}{\partial t}=-\frac{\partial}{\partial x}[\sqrt{2[E-\mathcal{P}(x)]}\,w]+2\frac{\partial}{\partial E}\left\{\left[\frac{1}{T}(E-\mathcal{P}(x))-\right.\right.$$
$$\left.\left.-\frac{1}{8}g(x)\right]w\right\}+\frac{1}{2}\frac{\partial^2}{\partial E^2}\{g(x)[E-\mathcal{P}(x)]w\}. \quad (3.79)$$

As before, us interests the solution of equation (3.79), close to the stationary. The two-dimensional density w(x, E) can be represented in the form

$$w(x,\ E)=w(x|E)\,w(E), \qquad (3.80)$$

where w(x|E) - the conditional density of distribution of value x.

For the particle, which moves in the potential field with small friction, the retention time in the vicinity of point x is inversely proportional to speed $\dot{x}=\sqrt{2[E-\mathcal{P}(x)]}$. Consequently,

$$w(x|E)=\begin{cases}\dfrac{C}{\sqrt{E-\mathcal{P}(x)}} & \text{при (1) } \mathcal{P}(x)<E,\\ 0 & \text{при (2) } \mathcal{P}(x)\geqslant E,\end{cases} \qquad (3.81)$$

Key: (1). with.

where C - coefficient, determined by standardization condition:

$$C=\left[\int_{R(E)}\frac{dx}{\sqrt{E-\mathcal{P}(x)}}\right]^{-1},$$

R(E) - range of values x, where $\mathcal{P}(x)<E$.

Let us substitute (3.80) in equation (3.79). Taking into account (3.81), let us produce the termwise integration of equation (3.79) for x in region R(E).

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As a result we will obtain the one-dimensional equation of Fokker-Planck relative to the density of distribution of energy w(E):

$$\frac{\partial w(E)}{\partial t}=\frac{\partial}{\partial E}\left\{\frac{w}{\varphi_2(E)}\left[\frac{\varphi_1(E)}{T}-\frac{\psi_2(E)}{2}\right]\right\}+\frac{1}{2}\frac{\partial^2}{\partial E^2}\left[w\frac{\psi_1(E)}{\varphi_2(E)}\right], \tag{3.82}$$

where

$$\varphi_1(E)=\int_{R(E)}\sqrt{E-\mathcal{P}(x)}\,dx,\quad \varphi_2(E)=\frac{1}{2}\int_{R(E)}\frac{dx}{\sqrt{E-\mathcal{P}(x)}};$$

$$\psi_1(E)=\frac{1}{2}\int_{R(E)}g(x)\sqrt{E-\mathcal{P}(x)}\,dx,$$

$$\psi_2(E)=\frac{1}{4}\int_{R(E)}\frac{g(x)\,dx}{\sqrt{E-\mathcal{P}(x)}},$$

moreover

$$\frac{d\varphi_1}{dE}=\varphi_2(E),\ \frac{d\psi_1}{dE}=\psi_2(E).$$

In the system with small friction (fading) the value of energy E is kept constant during several oscillatory periods. Therefore, if following error x at certain moment of time is within the limits of the aperture of discriminatory characteristic, but has the supply of energy E, greater than the height/altitude of potential threshold $\mathcal{P}_M$, then during the nearest period of oscillations x(t) will surmount this barrier and will achieve the absorbing boundary. Thus, a sufficient stall conditions of tracking is executing of the inequality

$$E > \mathcal{P}_M. \qquad (3.83)$$

In the general case in the servo system there are two potential thresholds (see Fig. 3.6); however, if the supply of energy E exceeds the height/altitude at least of smaller of them, then disruption/separation will occur with the probability, close to one.

Thus, the task about the disruption/separation of tracking in

the system of the second order is reduced to the solution of the one-dimensional equation of Fokker-Planck (3.82) with the boundary condition

$$w(E=\mathcal{P}_M)=0, \quad \mathcal{P}_M=\frac{K}{T}\int\limits_{x_A}^{x_2}[F(\zeta)-|A|]\,d\zeta, \qquad (3.84)$$

where x_2 - near to $\bar{x}_A$ point of unstable equilibrium.

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The problem, close to this, was examined during the analysis of the disruption/separation of tracking in first-order system; therefore let us now pause only at the separate stages of further solution.

The flow $\Pi(E)$ probability density, directed in the direction of the absorbing boundary, on the basis (3.82) is equal to

$$\Pi(E)=-\left[\frac{\varphi_1(E)}{T}-\frac{\psi_2(E)}{2}\right]\frac{w(E)}{\varphi_2(E)}-\frac{1}{2}\frac{\partial}{\partial E}\left[\frac{\psi_1(E)\,w(E)}{\varphi_2(E)}\right]. \qquad (3.85)$$

This expression can be converted [62] to the form

$$-\frac{\Pi(E)}{\psi_1(E)}e^{H(E)}=\frac{d}{dE}\left[\frac{w(E)}{2\varphi_2(E)}e^{H(E)}\right], \qquad (3.86)$$

where

$$H(E)=2\int\limits_0^E\frac{\varphi_1(\varepsilon)\,d\varepsilon}{T\psi_1(\varepsilon)}.$$

Considering flow as constant, let us integrate both parts of equality (3.86) with respect to E in the limits from zero to $\mathcal{P}_M$. Taking into account (3.84), we will obtain

$$\Pi = \frac{w(0)}{2\varphi_2(0)}\left[\int_0^{\mathcal{P}_M} \frac{\exp H(E)}{\psi_1(E)} dE\right]^{-1}. \qquad (3.87)$$

Let us expand function H(E) in power series in the vicinity of point $\mathcal{P}_M$. After taking having only given the first of term of expansion, let us compute the integral, entering expression (3.87). In this case we obtain $\psi_1(E) \approx \psi_1(\mathcal{P}_M)$ and let us replace lower integration limit by $-\infty$. After some conversions we will obtain

$$\Pi \approx \frac{w(0)}{\varphi_2(0)T}\varphi_1(\mathcal{P}_M)e^{-H(\mathcal{P}_M)}. \qquad (3.88)$$

Approximate value w(0) can be found from the steady-state solution of equation (3.85), which corresponds to $\Pi(E)=0$:

$$w(E) = C\varphi_2(E)\exp\left[-2\int_0^E \frac{\varphi_1(\varepsilon)}{T\psi_1(\varepsilon)} d\varepsilon\right]. \qquad (3.89)$$

Conclusions/outputs. The nonlinear system of the second order with the integrating filter in the general case to analyze difficultly. Expression (3.75) makes it possible to determine the probability of disruption/separation in the system under the effect on it of the noise whose spectral density does not depend on disagreement/mismatch x. The assumptions, done during conclusion/output (3.75), in essence are the same as they were accepted during the analysis of first-order systems. Calculation formula somewhat is simplified, if $KS_{*}T>>1$. In this case expression (3.76) for the probability of disrupting/separating the tracking in the nonlinear system coincides with the formula, obtained during the analysis of linear system, if the potential thresholds in both systems are identical.

The theoretical analysis of the disruption/separation of tracking in the systems where the level of noise effect depends on disagreement/mismatch x, is carried out only for the case of a small fading in the system. The probability of disruption/separation in this case is determined by dependence (3.90).

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3. System with astaticism of the second order.

Generalities. The tendency to decrease the dynamic errors in regulating circuits makes it necessary to use systems with the increased degree of astaticism. One of such systems, which obtained recently wide acceptance, is system with two integrators in the feedback loop and attenuating chain/network, which ensures the necessary stability factor. The resulting gear ratio/transmission factor of the feedback loop of this system is equal to

$$K(p)=\frac{K_2(1+pT_1)}{p^2}, \qquad (3.91)$$

and stochastic differential equation, describing the behavior of following error in the time, takes the form

$$\frac{d^2x}{dt^2}+KT_1\frac{dF(x)}{dx}\frac{dx}{dt}+KF(x)=$$
$$=\frac{d^2\lambda}{dt^2}-K\sqrt{N_0(x)}\xi^0(t)-KT_1\frac{d\,[\sqrt{N_0(x)}\xi^0(t)]}{dt}. \qquad (3.92)$$

The for the first time theoretical analysis of the disruption/separation of tracking in the system with astaticism of the second order was carried out by S. V. Pervachev in work [67]. Using basic ideas of this work, let us determine the probability of disruption/separation in the system in question during the smaller limitations to the form of input disturbances/perturbations, after placing $d^2\lambda/dt^2=\lambda_2=\text{const}\neq 0$ and taking into account the dependence of spectral density $N_0(x)$ on mismatch x.

As shown in [67] with sufficiently low value T_1 equation (3.92)

will approximately take the form

$$\frac{d^2x}{dt^2}+KT_1\frac{dF(x)}{dx}\frac{dx}{dt}+KF(x)-\lambda_2=K\sqrt{N_0(x)}\xi^0(t). \quad (3.93)$$

It describes the behavior of nonlinear system with a small changing in attenuation length. Introducing into the examination energy $E=\dot{x}^2/2+\mathcal{P}(x)$, let us represent equation (3.93) in the form of the system

$$\left.\begin{aligned}&\frac{dx}{dt}=\sqrt{2[E-\mathcal{P}(x)]},\\&\frac{dE}{dt}=-2[E-\mathcal{P}(x)]KT_1\frac{dF(x)}{dx}+\\&+\sqrt{2[E-\mathcal{P}(x)]}K\sqrt{N_0(x)}\xi^0(t),\end{aligned}\right\} \quad (3.94)$$

where $\mathcal{P}(x)$— potential energy

$$\mathcal{P}(x)=K\int_{x_A}^{x}[F(\zeta)-\Lambda]\,d\zeta; \quad (3.95)$$

$$\Lambda=\frac{\lambda_2}{K}.$$

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Probability of disruption/separation. System of equations (3.94) in form is analogous to system (3.78), which was examined during the analysis of follower with the integrating filter. Therefore, lowering conversions, identical to those carried out in the previous section, let us register the resultant expression for the probability of disruption/separation in the system with astaticism of the second order during the small fading:

$$P(l_{cp})\approx\frac{4K^2ST_1-1}{\pi g_3\sqrt{2KS}}KT_1\,[\text{illegible}](\mathcal{P}_M)\exp\left[-\frac{4T_1}{K}\int_0^{\mathcal{P}_M}\frac{\varphi(E)}{\psi(E)}dE\right], \quad (3.96)$$

where

$$\varphi(E) = \int_{R(E)} \sqrt{E - \mathcal{P}(x)}\, F'(x)\, dx; \qquad (3.97)$$

$$\psi(E) = \int_{R(E)} N_0(x) \sqrt{E - \mathcal{P}(x)}\, dx; \qquad (3.98)$$

$$F'(x) = \frac{dF(x)}{dx},\ S = F'(x_A),\ g_0 = \frac{K^2 N_0(x)}{2}\Big|_{x=0};$$

$$b = \frac{K^2}{4} \frac{d^2 N_0(x)}{dx^2}\Big|_{x=0};$$

$\mathcal{P}_M = \int_{x_A}^{x_B} K[F(x) - \Lambda]\, dx$ — the height/altitude of smaller potential threshold; x_A, x_B — respectively the coordinate of the points of stable and unstable equilibrium; R(E) - range of values x, where $E > \mathcal{P}(x)$.

In the particular case when spectral density is constant $N_0(x) = N_0$, formula (3.96) takes the form

$$P(t_n) \approx \frac{4ST_1^2 \sqrt{2KSt_n}}{\pi N_0} \varphi(\mathcal{P}_M) \exp\left[-\frac{4T_1}{K} \int_0^{\mathcal{P}_M} \frac{\varphi(E)}{\psi(E)} dE\right]. \qquad (3.99)$$

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Expression (3.99) is conveniently represented in the following form:

$$P(t_n) \approx \frac{4ST_1^2 \sqrt{2KSt_n}}{\pi N_0} \varphi(\mathcal{P}_M) \exp\left(-\varkappa^2 \frac{\gamma_0^2}{2\sigma_x^2}\right), \qquad (3.100)$$

where $\sigma_x^2 = \frac{N_0}{4} \frac{1 + KST_1^2}{S^2 T_1}$ - variance of error of tracking, found on the assumption that the servo system is linear, $\gamma_0 = \sqrt{\frac{2}{S} \int_{x_A}^{x_B} [F(x) - \Lambda]\, dx}$ —

equivalent threshold of the linear system, which has the same height/altitude of smaller potential threshold as initial nonlinear system; κ^2 - the correction factor, which considers the nonuniformity of friction in the system and equal to

$$\kappa^2 = \frac{N_0(1+KST_1^2)}{S\mathcal{F}_M}\int_0^{\mathcal{F}_M}\frac{\varphi(E)}{\psi(E)}\,dE \approx \frac{N_0}{S\mathcal{F}_M}\int_0^{\mathcal{F}_M}\frac{\varphi(E)}{\psi(E)}\,dE. \quad (3.101)$$

Recording (3.100) is convenient fact that the exponential member, who has the greatest effect on the value of the probability of disruption/separation, has the same form (with an accuracy to κ^2) as in expression (3.16), found according to the law of Poisson.

As showed the experimental check, carried out for the characteristics of the discriminators of different forms, expression (3.100) gives accuracy satisfactory for the practice at values of $KST^2{}_1 \leq 1$ (about 10-15% according to the relation of stresses/voltages signal/noise).

A special case. Let us give calculated relationships/ratios for the case when characteristic F(x) is approximated by the trapezoidal dependence (Fig. 3.8). The form of characteristic is determined by the slope/transconductance of the working section $S=F'(0)$ and by coefficient $\rho=(x^2-x_1)/x_1$. Let us assume the spectral noise density at the output of discriminator does not depend on disagreement/mismatch x, but input dynamic effect $\lambda(t)$ is such, that $d^2\lambda/dt^2=0$.

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Potential function in the system in question on the basis (3.95) is equal to

$$\mathcal{P}(x)=\begin{cases}\frac{KS}{2}x^2 & \text{при } |x|\leqslant x_1,\\ KSx_1|x|-\frac{KS}{2}x_1^2 & \text{при } x_1\leqslant|x|\leqslant x_2,\\ KSx_1x_2-\frac{KS}{2}(|x|-x_1-x_2)^2 & \text{при } x_2\leqslant|x|\leqslant x_1+x_2.\end{cases} \quad (3.102)$$

Key: (1). with.

The potential threshold which surmounts trajectory x(t) during the disruption/separation of tracking, it has the height

$$\mathcal{P}_M=\mathcal{P}(x_1+x_2)=KSx_1x_2.$$

The probability of disrupting/separating the tracking in the system in question is computed from formula (3.100), in which it is preliminarily necessary to determine $\varphi(\mathcal{P}_M)$ and $\varkappa$.

Let us find function $\varphi(\mathcal{P}_M)$. Taking into account (3.102), and also that

$$F'(x)=\frac{dF}{dx}=\begin{cases}S & \text{при } |x|\leqslant x_1,\\ 0 & \text{при } x_1\leqslant|x|\leqslant x_2,\\ -S & \text{при } x_2\leqslant|x|\leqslant x_1+x_2,\end{cases}$$

Key: (1). With.

we will obtain

$$\varphi(\mathcal{P}_M)=\int\limits_{-x_1-x_0}^{x_1+x_0}\sqrt{\mathcal{P}_M-\mathcal{P}(x)}\,F'(x)\,dx=$$

$$=S\sqrt{\frac{KS}{2}}\,x_1^2\left[\sqrt{2p+1}+2(p+1)\arcsin\sqrt{\frac{1}{2(p+1)}-1}\right].$$

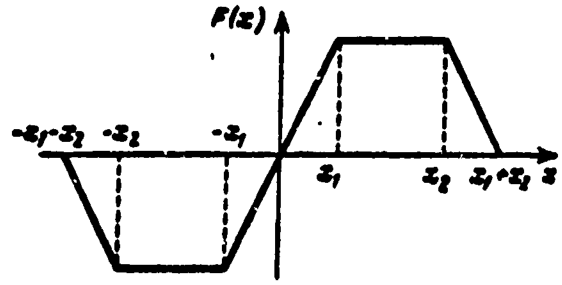

Fig. 3.8. Trapezoidal approximation of the characteristic of discriminator.

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Analogously we find

$$\frac{N_0\varphi(E)}{S\varphi(E)}=\begin{cases} 1 & \text{при } E<E_1, \\ \dfrac{e}{e+c} & \text{при } E_1<E<E_2, \\ \dfrac{e-d}{c+d+e} & \text{при } E_2<E<\mathcal{P}_м, \end{cases}$$

Key: (1). with.

where

$$c=\frac{2}{3}\left(\frac{E}{E_1}-1\right)^{3/2},\quad e=\sqrt{\frac{E}{E_1}-1}+\frac{E}{E_1}\arcsin\sqrt{\frac{E_1}{E}},$$

$$d=\sqrt{\frac{E-E_2}{E_1}}+\left(\frac{E-E_2}{E_1}-1\right)\frac{\ln\sqrt{1-\frac{E-E_2}{E_1}}}{1-\sqrt{\frac{E-E_2}{E_1}}},$$

$$E_1=\frac{KS}{2}x_1^2,\quad E_2=\frac{KS}{2}x_1^2(2\rho+1).$$

On the basis of the obtained relationships/ratios the

coefficient κ^2 can be calculated by the graphical integration. The dependence of the results of calculations on value ρ is depicted in Fig. 3.9 as solid line. Dotted line there constructed the dependence $\kappa(\rho)$, obtained experimentally. For the experimentation in the analog computer was gathered the ring of automatic control, described by differential equation (3.92).

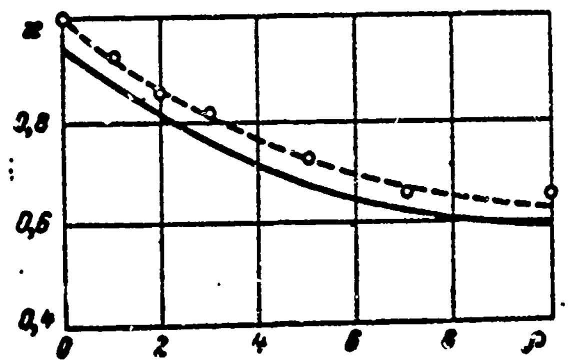

Fig. 3.9. Dependence of correction factor $\varkappa$ on the form of the characteristic of discriminator.

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To the entrance of system was supplied noise effect and by the repeated launchings/startings of machine was determined the probability of disrupting/separating the tracking [1].

FOOTNOTE [1]. In more detail the methodology of experiment in the analog computers is presented in S 6.1.

On the obtained probability with the help of relationship/ratio (3.100) was computed the corresponding value $\varkappa$. the disagreement between the theoretical and experimental values $\varkappa$ is caused by an error in the determination from formula (3.100) of the relation of stresses/voltages the signal/noise, with which the

disruption/separation of tracking occurs with the assigned probability. As can be seen from Fig. 3.9, this error does not exceed 10% over a wide range of a change in the form of the characteristic of discriminator.

As follows from (3.100), the greatest effect on the probability of disrupting/separating the tracking has the value of exponential term. Therefore for the approximate computation of probability the factor, which stands in formula (3.100) before the exponential curve, can be replaced with another expression by analogy with (3.7) so that the probability of disruption/separation would be equal to

$$P(t_n) \approx \frac{\omega_{11} t_n}{2\pi}\left[\exp\left(-x^2\frac{\gamma_{1\vartheta}^2}{2\sigma_x^2}\right)+\exp\left(-x^2\frac{\gamma_{2\vartheta}^2}{2\sigma_x^2}\right)\right], \quad (3.103)$$

where

$$\omega_{11}=\sqrt{KS\frac{(1+KST_1^2)^2}{1+(KST_1^2)^2}}$$ — the root-mean-square frequency of process x(t).

As shown in work [67], transition from formula (3.100) to (3.103) does not introduce into the calculation of appreciable error. Formula (3.103) it is possible to use with any attenuation lengths KST^2_1 in the system, selecting by correspondingly coefficient $\varkappa$. The theoretical analysis on the basis of which is constructed the dependence $\varkappa(\rho)$ in Fig. 3.9, was carried out on the assumption that the fading in the system is small ($KST^2_1 \ll 1$). The experimental check

showed that the obtained results can be used together with formula (3.103) up to values of $KST^2_1 \approx 1$. In this case an error in the determination of signal-to-noise ratio from the stress/voltage does not exceed 15%.

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With $KST^2_1 > 1$ the theoretical analysis of system to carry out is sufficiently difficult. For calculating the probability of disruption/separation in this case with the utilized trapezoidal approximation of characteristic F(x) it is possible to use formula (3.103), substituting in it the values $\varkappa$, found experimentally (Fig. 3.10). The graphs, constructed in Fig. 3.10, are described sufficiently well by the empirical formula

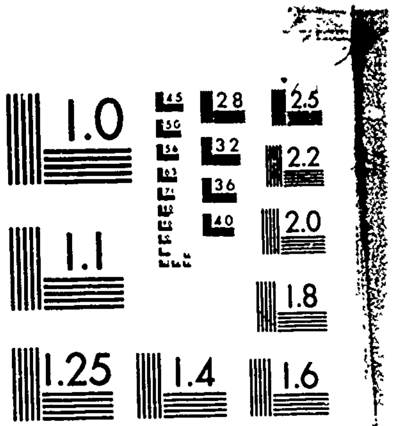
$$\varkappa \approx \frac{s(1+l)}{1+sl}, \qquad (3.104)$$

where

$$s = 0{,}5 + 0{,}33e^{-0{,}15\varphi};\ l = 0{,}18\,(1 + \lg KST_1^2)^2.$$

As can be seen from Fig. 3.10, with $KST^2_1 \geqslant 30$ value $\varkappa$ is virtually close to one with any form of the characteristic of discriminator. This is explained by the fact that with the the large KTS^2_1 the servo system in question is degenerated into first-order system, the probability of disruption/separation in which, as shown in this paragraph, is determined in essence by the height/altitude of

the potential threshold and virtually does not depend on the form of discriminatory characteristic.



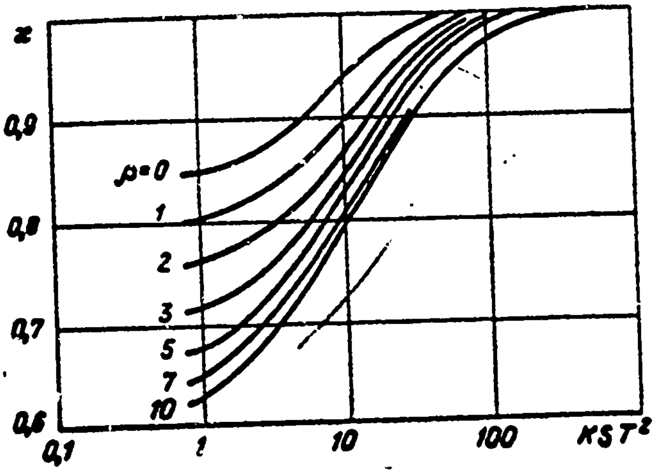


Fig. 3.10. Experimental values ϰ for the system with the high fading.

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Conclusions/outputs. For the system with astaticism of the second order in the arbitrary parameters of filter the calculation of the probability of disruption/separation can be carried out through approximation formula (3.103). When the analyzed system has small fading ($KST^2{}_1<1$), can be used more precise dependence (3.100). Formula (3.96) makes it possible to lead calculation taking into account the dependence of spectral noise density on disagreement/mismatch x.

When the linear section of the characteristic of discriminator F(x) comprises more than 1/3 apertures, coefficient ϰ differs from

one less than by 15%, independent of the parameters of system. This speaks, that the analysis of the disruption/separation of tracking in such systems with the small error can be produced by the methods of the theory of the ejections (see S 3.1) during the replacement of objective parameter F(x) of linear with the equivalent thresholds, determined by relationships/ratios (3.21).

For the characteristics with a small linear section a reduction in the coefficient κ depending on the parameters of system can be very essential. This it is necessary to consider during the identification of the parameters of system, which ensure the minimum probability of disrupting/separating the tracking, and at the determination of the required signal-to-noise ratio at the output of discriminator.

4. System of the second order with proportional-integrating filter.

Let us consider servo system with the gear ratio/transmission factor of feedback loop

$$K(p)=\frac{K(1+pT_1)}{p(1+pT)}. \qquad (3.105)$$

System with this filter possesses a series/row of advantages in comparison with the system with the simple integrating filter: by increased pull-in range, by best transient process and so forth, etc.

The differential equation, which describes the behavior of the system of the second order with proportional-integrating filter (3.105), takes the form

$$T\frac{d^2x}{dt^2}+\left(1+KTn\frac{dF(x)}{dx}\right)\frac{dx}{dt}+KF(x)=T\frac{d^2\lambda}{dt^2}+\frac{d\lambda}{dt}-$$
$$-K\sqrt{N_0(x)}\xi^0(t)-KTn\frac{d[\sqrt{N_0(x)}\xi^0(t)]}{dt}, \quad (3.106)$$

where $n=T_1/T$.

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This equation can be represented in the form of the system of two differential first-order equations (2.21). If we the coefficients of removal/drift write/record in the form, proposed by R. L. Stratonovich, the equation of Fokker-Planck will take form (2.38).

Let us consider the case when $N_0(x)=N_0=\mathrm{const}$, $d\lambda/dt=0$. By analogy with the previous calculation let us determine the probability of disrupting/separating the tracking in the system being investigated by expression (3.103), after changing by correspondingly of value ω_{II}, σ^2_φ and κ^2. Since ω_{II} and σ^2_φ are respectively root-mean-square frequency and dispersion in the linear system, on the basis (3.16) and (3.7) let us register

$$\omega_{II}=\sqrt{\frac{KS}{T}\frac{(1-n-KSTn^2)^2+KSTn^2}{1+KSTn^2}},$$

$$\sigma^2_\varphi=\frac{KN_0}{4S}\frac{1+KSTn^2}{1+KSTn}. \quad (3.107)$$

Let us determine coefficient $\varkappa$. With n→1 or T→∞ with n=const the analyzed regulating circuit is degenerated into first-order system with the coefficient it is born into first-order system with the gear ratio/transmission factor of feedback loop K(p) =Kn/p. A similar system is analyzed in p. 1 of this paragraph, whence it follows that in this case $\varkappa$=1. With n→0 the filter becomes integrating and the analysis, carried out in p. 2, shows that $\varkappa$=1. Determination $\varkappa$ in the arbitrary parameters of filter is connected with the considerable mathematical difficulties. Let us consider the special case when fading in the system is small (KST>1, KSTn²<0.5) [62].

With sufficiently low value KSTn²<<1 and $N_*(x)=N_0$ jamming intensity, as can be seen from (2.21), virtually it is possible to count independent from disagreement/mismatch x. This makes it possible to register initial differential equation (3.106) in the form

$$T\frac{d^2x}{dt^2}+\left[1+KTn\frac{dF(x)}{dx}\right]\frac{dx}{dt}+KF(x)=\xi_*(t), \quad (3.108)$$

where $\xi_*(t)$ — equivalent broadband noise. The intensity of this noise it is possible not to make more precise, since coefficient $\varkappa$ is determined not by the power of interference, but by the inconstancy of friction in the system.

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In view of the analogy between equations (3.108) and (3.93) the analysis of system with proportional-integrating filter can be carried out employing the procedure, which was being applied in the examination of system with astaticism of the second order. Lowering conversions, let us register final expression for determining the correction factor $\varkappa$:

$$\varkappa^2=\frac{1}{1+KSTn}\left[1+\frac{KTn}{\mathcal{F}_M}\int_0^{\mathcal{F}_M}\frac{\varphi(E)}{\psi(E)}\,dE\right], \qquad (3.109)$$

where

$$\left.\begin{aligned}&\varphi(E)=\int_{R(E)}\sqrt{E-\mathcal{F}(x)}\,\frac{dF(x)}{dx}\,dx;\\ &\psi(E)=\int_{R(E)}\sqrt{E-\mathcal{F}(x)}\,dx;\quad \mathcal{F}(x)=\frac{K}{T}\int_0^x F(\zeta)\,d\zeta;\end{aligned}\right\}\qquad (3.110)$$

$\mathcal{F}_M=\mathcal{F}(\gamma)$, γ — the boundary of the region of tracking; R(E) - range of values x, where $E\geq\mathcal{F}(x)$.

Comparing expressions (3.109) and (3.101), it is not difficult to be convinced of the existence of single-valued connection/communication of coefficient $\varkappa$ for the proportional-integrating filter with the analogous coefficient, found for the system with astaticism of the second order at the identical characteristics F(x) and small fading in both systems. Thus, the

calculation of the probability of disruption/separation in the system with the proportional-integrating filter substantially is facilitated, if with the same characteristic F(x) is known value κ in the system with astaticism of the second order. Thus, in the case of the trapezoidal characteristic F(x), depicted in Fig. 3.8, correction factor κ for the system with filter (3.105) can be determined via the corresponding recalculation of the graph $\kappa(\rho)$, constructed in Fig. 3.9.

Conclusions/outputs. The probability of disruption/separation in the system of the second order with the proportional-integrating filter is approximately determined by formula (3.103) during the appropriate replacement of entering it parameters ω_{II}, σ_e^2 and κ^2. Root-mean-square frequency ω_{II} and dispersion σ_e^2 of following error are computed with the help of approximate relationships/ratios (3.107).

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The correction factor κ, which considers the nonuniformity of fading in the system, during a small fading (KST>1, KSTn²<0.5) can be calculated by formulas (3.109)-(3.110).

On the accuracy of formula (3.103) for the system with the

proportional-integrating filter it is possible to judge by the family of curves (Fig. 3.11), obtained in work [62]. Solid lines in the figure correspond to the values κ, found theoretically with the help of relationship/ratio(3.109) at the trapezoidal characteristic F(x), which has linear section 21 times of less than the aperture of discriminator.

Are there constructed the dependences $\kappa(n)$, found by the simulation of system on the analog computer. As can be seen from figure, with the execution of conditions $KSTn^2<0.5$ and $KST>1$ the coefficient κ with an accuracy to 5-10% is determined by dependence (3.109). Is the same accuracy of the determination of the relation of stresses/voltages signal/noise on the output of discriminator with the assigned probability of disruption/separation $P\leqslant0.2$. As in the system with astaticism of the second order, coefficient κ differs from one not more than by 10%, if the linear section of the characteristic of discriminator composes at least 1/3 apertures.

From the analysis conducted it follows that in the systems with the proportional-integrating filter the coefficient κ is reduced most strongly with $KSTn^2\approx0.5$. This is explained by the fact that during this combination of the parameters the fading in the system, remaining small, considerably is changed due to the inconstancy of value $KnT(dF(x)/dx)$.

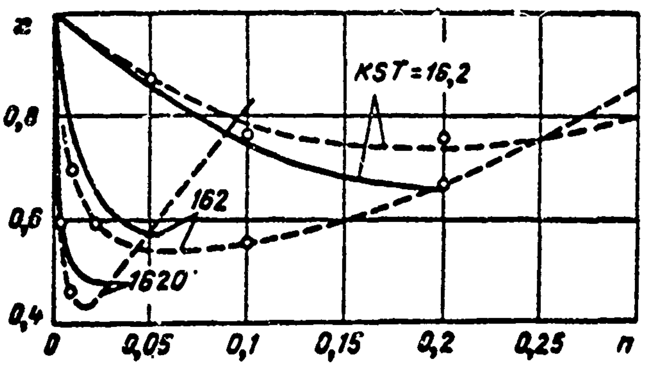

Fig. 3.11. Dependence of correction factor on the parameters of filter.

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A change of equivalent threshold $\varkappa\gamma_0$ in the system leads to the fact that value $n_{опт}$, which corresponds to the minimum probability of disruption/separation, does not coincide with value $n_{н}=1/\sqrt{KST}$, at which the variance of error of tracking in the linear system is minimum.

3.3. Special features/peculiarities of the analysis of systems with the periodic characteristics of discriminators.

A series/row of radio engineering systems of automatic tracking, such, for example, as the system of phase automatic frequency control (FAPCh), have the periodic characteristics of discriminators F(x). In

such systems are absent or are very small the ranges of values x, in which is not exhibited the controlling action of discriminator. Therefore in the general case it is difficult to unambiguously [illegible] boundaries γ_1 and γ_2 to the region of the trackings, output be[illegible] limits of which is identical to the disruption/separation of tracking.

The action of fluctuating interference in the systems with the periodic characteristics F(x) can lead to jumping over of following error to an arbitrary number of periods. The level of hazard of such migrations/jumps is determined by the conditions for work and by the concrete/specific/actual designation/purpose of the system of tracking. There are systems (for example, the tracking meters phases), for which a change in the following error for one period is inadmissible. In these cases it suffices to examine the behavior of process x(t) only in one period of discriminatory characteristic, considering that on its ends/leads are arranged/located the absorbing boundaries. The analysis of disruption/separation in this case is reduced to the already examined cases with the noncyclic characteristics F(x). By an example of the analysis of the system of phase automatic frequency control and as the integrating filter serves work [63].

However, there are many systems work of which does not

significantly affect transition x(t) to one or several periods. For example, this is the same system of phase self-alignment, which uses for the tracking the frequency of received signal (Fig. 3.12). In this device/equipment the single migration/jump of phase for the period yet does not lead to the loss of tracking the signal frequency.

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Let us consider, to what it leads noise effect on the system with the periodic characteristic F(x) based on the example of phase automatic frequency control. Despite the fact that to research of work of FAPCh during the noise effect are devoted numerous works [45, 51-54, 63, 70, 76-80, 83, etc.], the problem of the analysis of disruption/separation in this system is far not completely solved. However, the conducted investigations makes it possible to do a series/row of practically important conclusions/outputs and to give in certain cases the quantitative estimation of the degree of interference effect on the mode/conditions of tracking the frequency.

Noise effect on the system with the periodic characteristic F (x) to a considerable degree is determined by the presence of dynamic error and by the inertness of system. Dynamic error of FAPCh is characterized by initial detuning between the signal frequency and

natural frequency of the adjustable/tuneable generator. If detuning is more than pull-in range system of FAPCh, then even after the single migration/jump of phase error for the period the mode/conditions of tracking is broken with the probability, close to one. The same occurs, also, with small detuning, if the inertness of system is great. In these cases the single migration/jump of phase error for the period virtually leads to the disruption/separation of tracking the frequency; therefore during the analysis has the capability to place at the points of unstable equilibrium the absorbing boundaries.

In the remaining cases in the system of FAPCh are . established/installed the mode/conditions of tracking, from time to time interrupted/broken by separate short duration failures. If the frequency of the migrations/jumps of phase is small, this mode/conditions can prove to be permissible. It they frequently call the mode/conditions of asynchronous tracking [53].

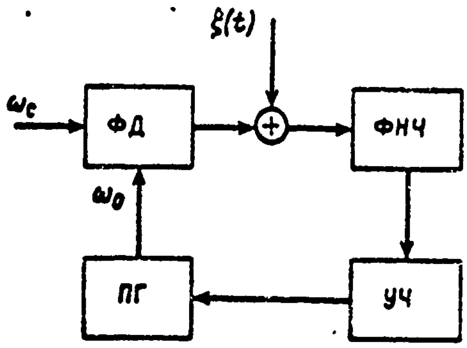

Fig. 3.12. Functional diagram of FAPCh: FD - phase discriminator; FNCh - low-pass filter; PG - readjustable generator; UCh - control of frequency.

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With the coincidence of the initial signal frequencies and adjustable/tuneable generator the medium frequency of the adjustable/tuneable generator will not be changed, since in this case the ejections of phase to the positive and negative sides are equiprobable. However, this fact yet does not make it possible to judge the reliability of the mode/conditions of tracking, since the dispersion of the frequency of the adjustable/tuneable generator in this case is different from zero.

If there is an initial detuning between the signal frequencies and adjustable/tuneable generator $\Delta\omega\neq0$, then the heights/altitudes of

two adjacent potential thresholds are not equal; therefore representative point attempts to roll down in the direction of smaller barriers. In this case together with the dispersion appears the constant component of frequency disagreement/mismatch.

Certain representation about the reliability of the mode/conditions of asynchronous tracking in the system FAPCh they can give average/mean value and the dispersion of frequency disagreement/mismatch and, especially, an average number of migrations/jumps of phase per unit time. Following works [47, 53, 64], let us determine these characteristics in the system FAPCh with the integrating filter. The differential equation of the system in question during the noise effect takes the form

$$T\frac{d^2\varphi}{dt^2}+\frac{d\varphi}{dt}+KF(\varphi)=\Delta\omega_0+K\sqrt{N_0}\xi^0(t), \qquad (3.111)$$

where T - time constant of the integrating filter; $\Delta\omega_0$ - initial detuning of the signal frequencies and adjustable/tuneable generator; N_0 - the spectral density of the broadband noise, led to the output of discriminator; $\xi^0(t)$ - single white noise; $F(\varphi)$ - the discriminatory characteristic of phase discriminator; K - gear ratio/transmission factor of the element/cell, which manages the frequency of the adjustable/tuneable generator.

Assuming/setting the characteristic of discriminator sinusoidal

$F(\varphi)$=U sin φ and introducing designations $\alpha=1/T$, $\Delta=KU$, $\Delta_0=\Delta\omega_0$, let us compose the equation of Fokker-Planck for the stationary two-dimensional probability density of combined phase distribution φ and difference frequency $\dot{\varphi}$:

$$\frac{B}{2}\frac{\partial^2 w(\varphi,\dot{\varphi})}{\partial\dot{\varphi}^2}=\frac{\partial}{\partial\dot{\varphi}}\left[\alpha(\Delta_0-\dot{\varphi}-\Delta\sin\varphi)w\right]+\dot{\varphi}\frac{\partial w}{\partial\varphi}, \quad (3.112)$$

where

$$B=\frac{\alpha^2K^2N_0}{2}.$$

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Since system is intended for the tracking the frequency, then it is unimportant, in what period of the characteristic of phase discriminator F(x) is realized this tracking. Consequently, during the determination of steady-state solution of $w(\varphi, \dot{\varphi})$ it is possible to use periodicity condition

$$w(\varphi,\dot{\varphi})=w(\varphi+2\pi,\dot{\varphi}). \quad (3.113)$$

Furthermore, function $w(\varphi, \dot{\varphi})$ must satisfy the condition for the standardization

$$\int_{-\pi}^{\pi}d\varphi\int_{-\infty}^{\infty}w(\varphi,\dot{\varphi})\,d\dot{\varphi}=1. \quad (3.114)$$

The exact solution of the steady-state equation of Fokker-Planck (3.112) taking into account conditions (3.113) and (3.114) in the absence of initial detuning (Δ_0=0) takes form [47]

$$w(\varphi,\dot{\varphi})=\sqrt{\frac{\alpha}{\pi B}}\frac{1}{2\pi I_0\left(\frac{2\alpha^2\Delta}{B}\right)}\exp\left[-\frac{\alpha\dot{\varphi}^2}{B}+\frac{2\alpha^2\Delta}{B}\cos\varphi\right]. \quad (3.115)$$

Hence it follows that the one-dimensional probability density of distributing the difference frequency $\dot{\varphi}$ is determined by the expression

$$w(\dot{\varphi})=\int_{-\infty}^{\infty} w(\varphi,\dot{\varphi})\,d\varphi=\sqrt{\frac{\alpha}{\pi B}}\exp\left(-\frac{\alpha\dot{\varphi}^2}{B}\right). \quad (3.116)$$

From (3.116) it follows that a difference in the signal frequencies and adjustable/tuneable generator in the absence of detuning is subordinated to the normal law of distribution with the zero average/mean value

$$\bar{\dot{\varphi}}=\int_{-\infty}^{\infty}\dot{\varphi}w(\dot{\varphi})\,d\dot{\varphi}=0 \quad (3.117)$$

and by the dispersion

$$\sigma_{\dot{\varphi}}^2=\frac{B}{2\alpha}. \quad (3.118)$$

If the initial detuning of frequencies $\Delta\omega_0$ is different from zero, to accurately solve equation (3.112) is sufficiently difficult.

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For the determination of approximate solution we will use the method, proposed by V. I. Tikhonov [47].

Let us determine the solution of equation (3.112) in the form of

the series/row

$$w(\varphi, \dot{\varphi}) = \sum_{n=0}^{\infty} L_n(\varphi)\, w^{(n)}(\dot{\varphi}), \qquad (3.119)$$

where function $L_n(\varphi)$ they are subject to definition, and

$$w(\dot{\varphi}) = \exp\left(-\frac{\alpha\dot{\varphi}^2}{B}\right). \qquad (3.120)$$

Let us substitute series/row (3.119) into the initial equation of Fokker-Planck (3.112) and will take into account that on the basis (3.120)

$$\dot{\varphi} w^{(n)}(\dot{\varphi}) = -\frac{B}{2\alpha} w^{(n+1)}(\dot{\varphi}) - n w^{(n-1)}(\dot{\varphi}). \qquad (3.121)$$

As a result we will obtain

$$\sum_{n=0}^{\infty} w^{(n+1)}(\dot{\varphi}) L'_n(\varphi) - \frac{2\alpha^2}{B}(\Delta_0 - \Delta\sin\varphi) \sum_{n=0}^{\infty} w^{(n+1)}(\dot{\varphi}) L_n(\varphi) =$$
$$= \frac{2\alpha}{B} \sum_{n=0}^{\infty} n w^{(n)}(\dot{\varphi}) L_n(\varphi) - \frac{2\alpha}{B} \sum_{n=0}^{\infty} n w^{(n-1)}(\dot{\varphi}) L'_n(\varphi). \qquad (3.122)$$

Equalizing coefficients with identical derivatives $w^{(n)}(\dot{\varphi})$, we will obtain the system of ordinary differential equations for determining the functions $L_n(\varphi)$:

$$\left.\begin{aligned} &L'_1(\varphi) = 0, \\ &L'_{n-1}(\varphi) - \frac{2\alpha^2}{B}(\Delta_0 - \Delta\sin\varphi) L_{n-1}(\varphi) = \\ &= 2n\frac{\alpha^2}{B} L'_n(\varphi) - 2(n+1)\frac{\alpha}{B} L'_{n+1}(\varphi). \end{aligned}\right\} \qquad (3.123)$$

Being limited by two members of sum of expression (3.119), let us register approximate solution of the equation of Fokker-Planck,

found taking into account periodicity conditions (3.113) and standardization (3.114)

$$w(\varphi, \dot{\varphi}) = w(\dot{\varphi}) L_0(\varphi) + w'(\dot{\varphi}) L_1(\varphi) =$$
$$= \sqrt{\frac{1}{\pi a B}} \frac{1-\exp(2\pi D_0)}{4\pi^2 \exp(\pi D_0)} |I_{iD_0}(D)|^{-2} \exp\left(-\frac{a\dot{\varphi}^2}{B}\right) \times$$
$$\times \left[-\dot{\varphi} + a \frac{\exp(2\pi D_0)}{1-\exp(2\pi D_0)} \exp(D_0\varphi + D\cos\varphi) \int_{\varphi}^{\varphi+2\pi} \exp(-D_0\gamma - \right.$$
$$\left. - D\cos\gamma)\, d\gamma \right], \qquad (3.124)$$

where $D_0 = \frac{2a^2\Delta_0}{B}$; $D = \frac{2a^2\Delta}{B}$; $I_{i\nu}(z)$ - function of Bessel of alleged index and alleged argument.

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Hence it follows that the one-dimensional densities of distribution of a phase difference and frequency take the form

$$w(\varphi) = \frac{1}{4\pi^2} \exp(\pi D_0) |I_{iD_0}(D)|^{-2} \exp(D_0\varphi + D\cos\varphi) \times$$
$$\times \int_{\varphi}^{\varphi+2\pi} \exp(-D_0\gamma - D\cos\gamma)\, d\gamma, \qquad (3.125)$$

$$w(\dot{\varphi}) = \left\{ \sqrt{\frac{a}{\pi B}} - \dot{\varphi} \sqrt{\frac{4\pi}{aB}} \frac{1-\exp(2\pi D_0)}{4\pi^2 \exp(\pi D_0)} \times \right.$$
$$\left. \times |I_{iD_0}(D)|^{-2} \right\} \exp\left(-\frac{a\dot{\varphi}^2}{B}\right). \qquad (3.126)$$

From (3.126) we find the average/mean detuning of the frequencies

$$\bar{\dot{\varphi}} = \Delta_0 \frac{\operatorname{sh} \pi D_0}{\pi D_0} |I_{iD_0}(D)|^{-2} \qquad (3.127)$$

and the dispersion

$$\sigma^2_{\dot{\varphi}} \approx \frac{B}{2a}. \qquad (3.128)$$

For evaluating the reliability of tracking FAPCh the frequency of received signal it is possible to introduce into the examination probability that an absolute difference in the frequencies will not exceed the allowed value of $\dot{\varphi}_0$:

$$P(|\dot{\varphi}|<\dot{\varphi}_0)=1-\int_{-\dot{\varphi}_0}^{\dot{\varphi}_0} w(\dot{\varphi})\,d\dot{\varphi}. \qquad (3.129)$$

Estimation only according to the average value $\overline{\dot{\varphi}}$ is insufficient, since with the zero detuning $\Delta\omega_0=0$ condition $|\overline{\dot{\varphi}}|<\dot{\varphi}_0$ is satisfied with any noise level and any $\dot{\varphi}_0$.

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The reliability of the mode/conditions of tracking can be judged also from the medium frequency of the migrations/jumps of phase. As is known [17], the frequency of the ejections of random function $\varphi(t)$ for the level Φ is determined by the formula

$$\nu(\Phi)=\int_0^{\infty}\dot{\varphi}w(\varphi=\Phi,\ \dot{\varphi})\,d\dot{\varphi}. \qquad (3.130)$$

Substituting in integral (3.130) the obtained expression for two-dimensional probability (3.124) and producing integration, we will obtain [64]

$$\nu(\Phi)=\sqrt{\frac{B}{4\pi\alpha}}\left[w(\Phi)+\frac{1}{4\alpha}\sqrt{\pi B}\,\frac{\operatorname{sh}\pi D_0}{\pi^2|I_{iD_0}(D)|^2}\right], \qquad (3.131)$$

where $w(\Phi)$ —one-dimensional density of distribution of phase, calculated according to formula (3.125) at point $\varphi=\Phi$. In the absence of initial detuning ($\Delta\omega_0=0$) expression (3.131) considerably is simplified

$$\nu(\Phi)=\sqrt{\frac{B}{\pi\alpha}}\frac{e^{D\cos\Phi}}{4\pi I_0(D)}. \qquad (3.132)$$

In particular, the frequency of the migrations/jumps of phase for levels $\Phi=\pm\pi$ is equal to

$$\nu(\pm\pi)=\sqrt{\frac{B}{\pi\alpha}}\frac{e^{-D}}{4\pi I_0(D)}. \qquad (3.133)$$

The method of determining the statistical characteristics of FAPCh examined can be, apparently, spred also to the systems with other filters. Such attempts are done in works [70, 78].

In [78] this method is used for the definition of characteristics of FAPCh with proportional-integrating filter. However, due to method accepted in this work of the expression of phase error $\varphi(t)$ through the components of two-dimensional Markov process, the obtained results are valid only in some special cases.

In work [70] is done the attempt to determine statistical characteristics of FAPCh with the arbitrary filter in the feedback

loop. However, the insufficient proof of some positions, which lie at the basis of the method proposed, requires the careful use/application of the obtained results.

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Relationships/ratios (3.131)-(3.133) make it possible to determine an average number of migrations/jumps of phase per unit time. However, under the noise effect are possible the migrations/jumps of the various kinds: to one, two or several periods for a comparatively short time, i.e., simultaneously can occur the series of migrations/jumps [56, 64]. On the duration of this series (about the number of periods to which they will be completed the migration/jump) the carried out analysis cannot give response/answer, since during the determination of the two-dimensional density of distribution of the probability of phase and frequency was used periodicity condition (3.113) of the solution of the equation of Fokker-Planck and thereby was carried out the averaging of statistical characteristics on all periods of characteristic $F(x)$. The attempt to approximately determine the distribution of migrations/jumps according to a number of periods is done in work [79] to the example to linearized FAPCh with the integrating filter. However, a quantity of assumptions done during the analysis and unwieldiness of final results impede the use of the latter in the

practice. Nevertheless, the series/row of the conclusions/outputs, obtained in [79], qualitatively correctly reflects physics of phenomenon.

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Chapter 4.

DISRUPTION OF TRACKING IN TIME-DEPENDENT SYSTEMS.

In the practice frequently it is necessary to deal concerning the regulating circuits processes in which carry unsteady character. The transiency of processes can be caused by the series/row of reasons. For example, dynamic disturbance/perturbation $\lambda(t)$ frequently leads to the fact that the mathematical expectation of following error $m_x(t)$ becomes the function of time. Transiency can be the corollary of inconstancy in the time of noise level $\xi(t)$. With it it is necessary to be counted, if the time of the establishment of transient processes in the system is commensurated with the time of observation. Therefore the analysis of the disruption/separation of tracking in the time-dependent systems represents urgent task. In this chapter are examined several approximation methods, which allow with one or the other degree of accuracy to take into account the transiency of processes during the analysis of the disruption/separation of tracking regulating circuits.

4.1. Generalization of the theory of ejections for the analysis of time-dependent systems.

Generalization of Poisson's law. The simplest methods of the approximate definition of the probability of disruption/separation, as this follows from the previous material, gives the theory of the ejections of random functions, which uses a Poisson character of the distribution of the rare ejections of noise. However, during the derivation of formula (3.4) it was assumed that the frequency of the ejections of process x(t) above the level γ does not change in time. In the time-dependent systems this condition is not satisfied, since either the dispersion of process $\sigma^2_x(t)$, or the level of equivalent $\gamma_0(t)=\gamma-m_x(t)$ threshold is the function of time.

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For the analysis of such systems with the help of the theory of ejections let us introduce the function $\nu(t)$, which characterizes the frequency of the ejections of the unsteady process x(t) above the variable/alternating threshold $\gamma(t)$. Let us assume that the noise level is sufficiently small so that the separate ejections for the threshold would prove to be independent variables. Let us decompose

entire time of observation t_{H} to m of sufficiently small sections with the duration Δt so that in each i section the frequency of ejections it would be possible to consider constant $\nu_i=\nu(t_i)$.

Taking into account the mutual independence of the ejections of noise above the level γ(t), let us determine the probability of the appearance at least of one ejection for time t_H:

$$P(t_H)=1-\prod_{i=1}^{m} e^{-\nu_i \Delta t}=1-\exp\left(-\sum_{i=1}^{m} \nu_i \Delta t\right).$$

Passing to the limit with Δt→0, we will obtain the generalization of Poisson's formula (3.2) to the unsteady case

$$P(t_H)=1-\exp\left(-\int_0^{t_H} \nu(t)\,dt\right). \qquad (4.1)$$

Thus, the probability of disruption/separation in the time-dependent system of automatic control on the sufficiently small noise level is approximately determined by the dependence

$$P(t_H)=1-\exp\left\{-\int_0^{t_H}[\nu_1(t)+\nu_2(t)]\,dt\right\}\approx$$

$$\approx\int_0^{t_H}[\nu_1(t)+\nu_2(t)]\,dt. \qquad (4.2)$$

Here in contrast to (4.1) is taken into consideration the presence of two thresholds γ_1(t) and γ_2(t), which respectively led to the appearance of two addend in the frequency disruptions/separations ν_1(t) and ν_2(t). Dependence (3.4) is obtained from (4.2) as partial

case when $\nu(t)$=const.

Frequency of ejections. Further calculation of the probability of disruption/separation requires the determination of the dependence of the frequency of ejections of $\nu_1(t)$ and $\nu_2(t)$ on the time and on the parameters of system.

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As shown in [17], frequency $\nu^+(t)$ of the intersection with random process of x(t) of the determined function $\gamma(t)$ from bottom to top [1] is determined by the dependence

$$\nu^+(t)=\int_0^\infty \dot\eta(t)\, w(\gamma(t),\ \dot\gamma(t)+\dot\eta(t))\, d\dot\eta,\quad \eta(t)=x(t)-\gamma(t),$$

where

$$w(\gamma(t),\ \dot\gamma(t)+\dot\eta(t))=w(x,\ \dot x)\Big|_{\substack{x=\gamma(t),\\ \dot x=\dot\gamma+\dot\eta}},$$

$w(x, \dot x)$ - the combined density of distribution of process of x(t) and its derivative.

FOOTNOTE [1]. By intersection from bottom to top is understood the event, consisting in the fact that the sign of expression $\gamma(t)$-x(t) varies from the positive to the negative. The value of the derivative of random process x(t) in this case can be negative. ENDFOOTNOTE.

It is analogous, for the frequency $\nu^-(t)$ ejections downward we have

$$\nu^-(t) = -\int_{-\infty}^{0} \dot{\eta}(t)\, w(\gamma(t), \dot{\gamma}(t) + \dot{\eta}(t))\, d\dot{\eta}.$$

Thus, the frequency of disruptions/separations in the system with two boundaries is defined as

$$\nu_1(t) + \nu_2(t) = \int_{0}^{\infty} \dot{\eta}_2(t)\, w(\gamma_2(t), \dot{\gamma}_2(t) + \dot{\eta}_2(t))\, d\dot{\eta}_2 -$$
$$- \int_{-\infty}^{0} \dot{\eta}_1(t)\, w(\gamma_1(t), \dot{\gamma}_1(t) + \dot{\eta}_1(t))\, d\dot{\eta}_1, \qquad (4.3)$$

where

$$\eta_i = x(t) - \gamma_i(t).$$

The direct calculation of the frequency of disruptions/separations according to (4.3) is difficult, since usually the two-dimensional density $w(x, \dot{x})$ of distribution is unknown. Exception are the linear systems in which the following error is distributed according to the normal law. In the literature is described a series/row of special cases of the transiency of noise and threshold when it is possible to obtain exact expressions for the average number H(t) of ejections for the preset time of observation t [17, 22, etc.]. Since $\nu(t) = dH(t)/dt$, then these expressions can prove to be useful during the calculation of the probability of disruption/separation. In particular, from the mentioned works it follows that the frequency of the ejections of the normal stationary process x(t) above level $\gamma_0(t) = \gamma - s(t)$, where s(t) - the determined function, is determined by the expression

$$v(t)=\frac{\omega_{11}}{2\pi}\exp\left(-\frac{\gamma_0^2(t)}{2\sigma_x^2}\right)\left\{\exp\left(-\frac{\dot{s}^2(t)}{2\omega_{11}^2\sigma_x^2}\right)+\right.$$
$$\left.+\sqrt{\pi}\frac{\dot{s}(t)}{\sqrt{2}\omega_{11}\sigma_x}\left[1+\Phi\left(\frac{\dot{s}(t)}{\sqrt{2}\omega_{11}\sigma_x}\right)\right]\right\}, \qquad (4.4)$$

where $\dot{s}(t)$ =ds/dt; $\Phi(z)$ - probability integral (1.5); σ^2_x — the dispersion of process of x(t); ω_{11} — the root-mean-square frequency of process x(t), determined by relationship/ratio (3.8). During conclusion/output (4.4) it was assumed that process x(t) was centralized, i.e., $m_x(t)=0$.

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If rate of change in the time of threshold $\gamma_0(t)$ is small in comparison with the speed of process x(t), in other words, if $\dot{s}(t)/\omega_{11}\sigma_x \ll 1$, then expression in the curly braces of formula (4.4) approaches one, and formula takes the form

$$v(t)\approx\frac{\omega_{11}}{2\pi}\exp\left(-\frac{\gamma_0^2(t)}{2\sigma_x^2}\right). \qquad (4.5)$$

From the comparison of expressions (4.5) and (3.7) it is evident that in the case of the slow transiency of threshold during the calculation of the frequency of ejections it suffices in formula (3.7) to replace constant threshold γ with variable/alternating $\gamma_0(t)$. The analogous fact of

takes place also when normal process is unsteady, and rate

of change in its dispersion is small. In this case the frequency of ejections can be found according to formula (3.7), where should be taken into account change in the time of dispersion and threshold.

Taking into account that the calculation of the frequency of disruptions/separations in the nonlinear systems is hindered/hampered, for the proximate analysis of such systems is carried out their preliminary linearization. For this, as in the fixed systems, are applied the method of statistical linearization or linearization on the criterion of the equality of the supply of potential energy in the linear and nonlinear systems. It must be noted that this methodology of the determination of the probability of disruption/separation does not possess high accuracy.

Harmonic effect. In order to consider, in what cases during the analysis of the disruption/separation of tracking it is necessary to consider the dynamics of processes, let us consider regulating circuit (see Fig. 1.2), at the entrance of which functions the harmonic disturbance/perturbation:

$$\lambda(t)=\lambda_0 \sin \omega_0 t. \tag{4.6}$$

We will consider that the random effect $\xi(t)$ converted to the output of discriminator is white noise with a spectral density of N_0=const.

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For simplicity let us assume also, that the discriminatory characteristic is linear in the limits of the aperture

$$F(x)=Sx,\ -\gamma<x<\gamma.$$

Under the action of regular disturbance/perturbation (4.6) the mathematical expectation of following error in the steady-state mode/conditions varies in the time also according to the sinusoidal law. With an accuracy to phase it is possible to register

$$m_x(t)=A\sin\omega_0 t,$$

moreover amplitude A of oscillation is defined as

$$A=\lambda_0\left|\frac{1}{1+SK(j\omega)}\right|_{\omega=\omega_0},$$

where S - slope/transconductance of discriminatory characteristic; $K(j\omega)$ - the complex frequency characteristic of the feedback loop of system.

We centralize following error and will introduce the equivalent thresholds

$$\gamma_{10}=-\gamma-A\sin\omega_0 t,$$

$$\gamma_{20}=\gamma-A\sin\omega_0 t.$$

Assuming that the frequency of disturbance/perturbation ω_0 is considerably lower than the root-mean-square frequency ω_{II} of servo system, for calculating the probability of disruption/separation we

will use relationships/ratios (4.2) and (4.5). As a result we will obtain

$$P(t_n)=\frac{\omega_{II}}{2\pi}\left\{\int_0^{t_n}\exp\left[-\frac{\gamma^2}{2\sigma_x^2}\left(1+\frac{A}{\gamma}\sin\omega_0 t\right)^2\right]dt+\right.$$

$$\left.+\int_0^{t_n}\exp\left[-\frac{\gamma^2}{2\sigma_x^2}\left(1-\frac{A}{\gamma}\sin\omega_0 t\right)^2\right]dt\right\}. \qquad (4.7)$$

If $A/\gamma<0.25-0.2$, then without the large error it is possible to register

$$\left(1\pm\frac{A}{\gamma}\sin\omega_0 t\right)^2\approx 1\pm\frac{2A}{\gamma}\sin\omega_0 t.$$

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During the calculation of the integrals, entering expression (4.7), we use an expansion of exponential curves in the series/rows in the modified Bessel functions:

$$\left.\begin{aligned} e^{\alpha\sin x}&=I_0(\alpha)+2\sum_{k=1}^{\infty}I_k(\alpha)\cos k\left(\frac{\pi}{2}-x\right),\\ e^{-\alpha\sin x}&=I_0(\alpha)+2\sum_{k=1}^{\infty}(-1)^k I_k(\alpha)\cos k\left(\frac{\pi}{2}-x\right).\end{aligned}\right\} \qquad (4.8)$$

As a result for the probability of disruption/separation taking into account dynamics we have

$$P(t_n)=\frac{\omega_{II}}{\pi\omega_0}\exp\left(-\frac{\gamma^2}{2\sigma_x^2}\right)\left\{\omega_0 t_n I_0(\alpha)-\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{1}{k}I_{2k}(\alpha)\sin\left[k(\pi-2\omega_0 t_n)\right]\right\}. \qquad (4.9)$$

Coefficient $\alpha = A\gamma/\sigma_x^2$ characterizes the ratio of the intensity of dynamic effect to the aperture of discriminator and power of fluctuations.

Let us note that the probability of disruption/separation in the analyzed system, found without taking into account dynamic disturbance/perturbation, is determined by the dependence

$$P_0(t_n) = \frac{\omega_{11} t_n}{\pi} \exp\left(-\frac{\gamma^2}{2\sigma_x^2}\right).$$

To consider the degree of the effect of dynamic disturbance/perturbation on the probability of disruption/separation is possible, after calculating the relation of probabilities $P(t_n)/P_0(t_n)$. In particular, if the time of observation is multiple to half of the period of perturbing effect ($t_n\omega_0 = m\pi$), then

$$\frac{P(t_n)}{P_0(t_n)} = I_0(\alpha).$$

Table 4.1 gives the values of the relations of the probabilities of disruption/separation depending on values α, found taking into account the action of dynamic disturbance/perturbation λ(t), also, without it.

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From the table it is evident that during the analysis of the disruption/separation of tracking it is possible to disregard the action of harmonic disturbance/perturbation, if coefficient α<1-2. With α>3 ignoring the dynamics of process $\lambda(t)$ leads to the very large errors in the determination of the probability of disruption/separation.

To approximately take into account the unsteady dynamic effect is possible also by the replacement of real mathematical expectation $m_x(t)$ by its effective value. Thus, for the case of harmonic effect in question we approximately consider that the dynamic following error is time-constant and is equal to $m_x(t)=m_0=A/\sqrt{2}$. In this case, as it follows from § 3.1, the probability of disruption/separation is determined by the formula

$$P_{s\phi}(t_n)=\frac{\omega_{11}t_n}{2\pi}\left\{\exp\left[-\frac{(\gamma-m_0)^2}{2\sigma_x^2}\right]+\exp\left[-\frac{(\gamma+m_0)^2}{2\sigma_x^2}\right]\right\}.$$

Comparing the obtained result with a more precise (4.9) with $t_n=\frac{m\pi}{\omega_0}$, m=1, 2, 3, ..., let us find an error in this method of the account of the sinusoidal dynamic effect

$$\frac{P(t_n)}{P_{s\phi}(t_n)}=\frac{I_0(\alpha)}{\operatorname{ch}\frac{\alpha}{2}}e^{-A^2/4\sigma_x^2},\quad \alpha=\frac{A}{\gamma}\frac{\gamma^2}{\sigma_x^2}.$$

From the latter/last relationship/ratio it follows that the error in the determination of the probability of disruption/separation is less, the less the amplitude A of dynamic

error in comparison with the aperture 2γ of discriminatory characteristic and rms error σ_x.

4.2. Method of Bubnov - Galerkin.

The method of Bubnov - Galerkin [12] is the well developed method of approximate solution of the tasks of mathematical physics. For the first time the analysis of the disruption/separation of tracking by this method is carried out by I. A. Bol'shakov [46].

Table 4.1.

a	0	0,5	1	2	3	5
$P(t_{н})/P_0(t_{н})$	1	1,06	1,27	2,28	4,88	27,2

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In principle the method of Bubnov - Galerkin let us use for solving the equations of arbitrary dimensionality. However, unwieldiness of linings/calculations does not make it possible to use it for solving the multidimensional diffusion equations. In connection with this we will be bounded in this paragraph to the analysis first-order of servo systems.

Let the behavior of system be described stochastic differential equation

$$\frac{dx}{dt}=a(x, t)+b(x, t)\xi^0(t) \tag{4.10}$$

with the coefficients of a(x, t) and b(x, t) nonlinear in the general case. The value of following error at the moment of the beginning of observation t=0 let us designate x_0. The transiency of task depends on the dependence of coefficients a and b on the time and on the presence of the transient mode/conditions of the establishment of tracking error.

The probability of disruption/separation $P(x_0, t)$ in the system of tracking is determined as a result of solving the equation of Pontriagin (2.78), supplemented by initial (2.80) and boundary (2.84) conditions. In order to have uniform boundary conditions, let us introduce function $U(x_0, t)=1-P(x_0, t)$, which is the probability of retaining/preserving/maintaining the mode/conditions of tracking in the system for a period of time t, if at zero time following error was equal to x_0. As a result of this replacement we will obtain the following boundary-value problem:

$$\frac{\partial U(x_0, t)}{\partial t}=A(x_0, t)\frac{\partial U}{\partial x_0}+\frac{B(x_0, t)}{2}\frac{\partial^2 U}{\partial x_0^2}, \qquad (4.11)$$

$$U(x_0, 0)=1, \; \gamma_1<x_0<\gamma_2, \qquad (4.12)$$

$$U(\gamma_1, t)=U(\gamma_2, t)=0, \qquad (4.13)$$

where coefficients $A(x_0, t)$ and $B(x_0, t)$ are expressed as the coefficients of initial stochastic equation (4.10) [see (2.31)-(2.33)].

If in equation (4.11) coefficients A and B do not depend on time, then variable/alternating are divided. Actually/really, if we introduce

$$U(x_0, t)=T(t)X(x_0), \qquad (4.14)$$

that

$$\frac{T'(t)}{T(t)}=\frac{\frac{1}{2}B(x_0)X''(x_0)+A(x_0)X'(x_0)}{X(x_0)}=\lambda. \qquad (4.15)$$

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The exact solution of boundary-value problem for eigenfunctions $X(x_*)$ succeeds in finding very rarely. Exception is the task about the disruption/separation of tracking in the system from the linear in the limits aperture by the characteristic of discriminator ($A(x)=Sx$, $B(x)=const$) [49, 59, 65, 74, 75]. However, even in this simplest task $X(x_*)$ it is not expressed as elementary functions, which impedes calculation. For the analysis of nonlinear systems it is expedient to use the approximate methods of solving the boundary-value problems, one of which is the method of Bubnov - Galerkin.

The method of Bubnov - Galerkin, actually, is further development of the method of separation of variable/alternating. Because of the fact that as functions $X(x_*)$ are used not the eigenfunctions of differential equation (4.15), but the previously selected coordinate functions, the process of solving the boundary-value problem significantly is simplified and the class of tasks solved by this method can be expanded virtually for arbitrary $A(x, t)$ and $B(x, t)$.

Fundamental principles. According to the method of Bubnov - Galerkin the n approximation of solution of task (4.11)-(4.13) let us define as

$$U_n(x_0, t)=\sum_{k=1}^{n} C_k(t)\varphi_k(x_0), \qquad (4.16)$$

where $\{\varphi_k(x_0)\}$ - the complete in the region $\gamma_1 \leqslant x_0 \leqslant \gamma_2$ system of the coordinate functions, which are rotated into zero on the boundary of the region.

Factors C_k are determined from the condition of orthogonality $L[U_n(x_0, t)]$ to all functions $\varphi_k(x_0)$. Here L - differential partial differential operator, who corresponds to writing of equation (4.11) in the form L $[U(x_0, t)]=0$. In this case

$$L=\frac{\partial}{\partial t}-A(x_0, t)\frac{\partial}{\partial x_0}-\frac{1}{2}B(x_0, t)\frac{\partial^2}{\partial x_0^2}.$$

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The condition of orthogonality reduces to the system of the differential equations

$$(L[U_n(x_0, t)], \varphi_l)=0, \qquad (4.17)$$
$$l=1, 2, \ldots, n,$$

where (u, v) indicates the scalar product of functions u and v.

Functions $C_k(t)$, which depend only on the time, will be taken out as the sign of scalar product; therefore (4.17) it is system n first-order of ordinary differential equations relative to functions $C_k(t)$.

Initial conditions $C_k(0)$, necessary for the unique solution of system (4.17), are determined from the resolution of initial condition $U(x_0, 0)$ in the series/row in terms of the set of functions φ_k:

$$\sum_{k=1}^{\infty} C_k(0)\varphi_k(x_0) = U(x_0, 0) = 1. \qquad (4.18)$$

System of equations (4.17) considerably is simplified, if as the set of functions φ_k are selected the orthogonal functions

$$(\varphi_k, \varphi_l) = 0 \overset{(1)}{\text{при}}\ k \neq l. \qquad (4.19)$$

Key: (1). with.

In that case the system of differential equations (4.17) takes the form

$$\frac{dC_l(t)}{dt} = \frac{1}{(\varphi_l, \varphi_l)}\sum_{k=1}^{n} C_k(t)\left[\left(A(x_0, t)\frac{d\varphi_k(x_0)}{dx_0}, \varphi_l(x_0)\right) + \left(\frac{1}{2}B(x_0, t)\frac{d^2\varphi_k(x_0)}{dx_0^2}, \varphi_l(x_0)\right)\right], \quad l = 1, 2, \ldots, n. \qquad (4.20)$$

Using a property of orthogonality (4.19), from (4.18) we find

$$C_l(0) = \frac{(1, \varphi_l)}{(\varphi_l, \varphi_l)}. \qquad (4.21)$$

The scalar product of two functions let us define as the integral

$$(u, v)=\int_{\gamma_1}^{\gamma_2} \rho(x)\, u(x)\, v(x)\, dx, \qquad (4.22)$$

where $\rho(x)$ - weight function.

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The successful selection of weight function and sequence of coordinate functions $\varphi_k(x)$ in many respects contributes to the success in the solution of problem, raising the speed of the convergence of approximations/approaches $U_n(x_0, t)$.

First method. The simplest and spread method of the solution of problem presumes that as the coordinate functions are used the functions of the trigonometric series:

$$\varphi_1(x)=\cos\frac{\pi x}{2\gamma},\ \ \varphi_2(x)=\sin\frac{\pi x}{\gamma},\ \ \varphi_3(x)=\cos\frac{3\pi x}{2\gamma},$$
$$\varphi_4(x)=\sin\frac{2\pi x}{\gamma},\ldots \qquad (4.23)$$

with single weight function

$$\rho(x)=1. \qquad (4.24)$$

Here and subsequently for the convenience it is placed $\gamma_2=-\gamma_1=\gamma$, what always it is possible to attain by the replacement of the variable/alternating x_0.

For solving the system of equations (4.20) it is necessary to preliminarily compute the scalar products, entering the coefficients with unknowns $C_k(t)$. System (4.20) considerably is simplified, if coefficients A and B do not depend on the time: $A(x_0, t)=A(x_0)$, $B(x_0,t)=B(x_0)$. Then (4.20) it is the system of linear equations with the constant coefficients, and its solution is located analytically.

Second method. If the coefficients stochastic equation do not depend on time, then it is possible to attempt to improve the convergence of solution (4.16) by the selection of weight factor $\rho(x)$.

It is known [14] that eigenfunctions $X_k(x_0)$ of equation (4.11) are orthogonal with a weight of $w_{ст}(x)$:

$$\int_{-\gamma}^{\gamma} w_{ст}(x)\, X_k(x)\, X_l(x)\, dx = 0 \ \overset{(1)}{\text{при}}\ k \neq l. \qquad (4.25)$$

Key: (1). with.

Here $w_{ст}(x)$ — the solution of the steady-state equation of Fokker - Planck:

$$\frac{d}{dx}[A(x)\, w_{ст}(x)] = \frac{1}{2}\frac{d^2}{dx^2}[B(x)\, w_{ст}(x)], \qquad (4.26)$$

the supplemented by conditions reflection at points $=\gamma$:

$$\Pi(\gamma, t) = \Pi(-\gamma, t) = 0.$$

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The solution of equation (4.26) takes the form

$$w_{\text{ст}}(x)=\frac{c}{B(x)}\exp\left\{2\int_0^x\frac{A(\zeta)}{B(\zeta)}d\zeta\right\}, \qquad (4.27)$$

where c - constant, determined from standardization condition.

The orthogonality of eigenfunctions $X_k(x_0)$ with a weight of $w_{\text{ст}}(x_0)$ mak it possible to assume that in scalar product (4.22) as the weight factor it is expedient to select

$$\rho(x)=w_{\text{ст}}(x). \qquad (4.28)$$

The nearer the system of coordinate functions $\varphi_k(x_0)$ to eigenfunctions $X_k(x_0)$, the more precise approximate representation (4.16). Let us isolate in functions $\varphi_k(x_0)$ factor $\sqrt{w_{\text{ст}}(x_0)}$ and it is represented them in the form

$$\varphi_k(x_0)=[w_{\text{ст}}(x_0)]^{-1/2}\psi_k(x_0), \qquad (4.29)$$

where as $\psi_k(x_0)$ it is convenient to select the system of orthogonal funtions

$$\int_{-1}^{1}\psi_k(x)\psi_l(x)\,dx=0 \quad \overset{(1)}{\text{при}}\ k\neq l. \qquad (4.30)$$

Key: (1). with.

System of coordinate functions $\varphi_k(x_0)$ selected thus is orthogonal with a weight of $w_{\text{ст}}(x_0)$. As a result the system of equations (4.17) is converted to the form

$$\frac{dC_l(t)}{dt}=\frac{1}{2}\left[\int_{-\gamma}^{\gamma}\psi_l^2(x)\,dx\right]^{-1}\sum_{k=1}^{n}C_k(t)\int_{-\gamma}^{\gamma}\left\{B(x)\psi''_k(x)\psi_l(x)+\right.$$

$$+B'(x)\psi'_k(x)\psi_l(x)-\left[\frac{A^2(x)}{B(x)}+A'(x)+\right.$$

$$\left.\left.+\frac{B'(x)\,[B'(x)-4A(x)]}{4B(x)}-\frac{1}{2}B''(x)\right]\psi_k(x)\psi_l(x)\right\}dx, \quad (4.31)$$

$$l=1,2,\ldots,n.$$

The simplest orthogonal functions are trigonometric (4.23), which can be used as system $\psi_k(x)$.

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In order to explain advantages and deficiencies/lacks in the described methods of determining of scalar product and selection of the system of coordinate functions, let us solve by two methods the following task.

Example 1. We analyze the simplest (without the filter) system of phase automatic frequency control whose functional diagram is represented in Fig. 3.12. Considering that the control of the phase

of the adjustable/tuneable generator is produced via the rearrangement of its frequency, let us register differential equation of FAPCh:

$$\frac{dx}{dt} = \omega - \Omega F_{\bullet}(x) + K\sqrt{N_{\bullet}}\xi(t), \qquad (4.32)$$

where x - instantaneous phase difference of the oscillations of input signal and adjustable/tuneable generator; $F_{\bullet}(x)$ - calibrated ($\max F_{\bullet}(x)=1$) the characteristic of phase detector; $\omega=\omega_0-\omega_c$ — the initial detuning of the frequency of the adjustable/tuneable generator $\omega_{\bullet}$ relative to signal frequency ω_c; $\Omega=u_m K$ — the band of synchronism; u_m — the maximum stress/voltage, developed by phase discriminator; K - mutual conductance of the control of frequency; $N_{\bullet}$ - spectral noise density, led to the output of phase discriminator. Let us note that differential equation (4.32) under specific conditions approximately describes also system of FAPCh with the proportional-integrating filter.

The discriminatory characteristic of phase discriminator is frequently approximated by expression $F_{\bullet}(x)=\sin x$.

We will be bounded to the examination of the case $\omega=0$. For probability $U(x_{\bullet}, t)$ of absence for time t of the migrations/jumps of the phase through the points of unstable equilibrium $-\pi$ and π with initial disagreement/mismatch of phases, equal $x(0)=x_{\bullet}$, occurs the boundary-value problem

$$\frac{\partial U(x_0,\tau)}{\partial \tau} = -\alpha \sin x_0 \frac{\partial U}{\partial x_0} + \frac{\partial^2 U}{\partial x_0^2}, \qquad (4.33)$$

$$U(x_0, 0) = 1, \quad -\pi < x_0 < \pi, \qquad (4.34)$$

$$U(-\pi, \tau) = U(\pi, \tau) = 0, \qquad (4.35)$$

where $\alpha=4\Omega/K^2N_0$ - ratio of the power of signal and noise in the band of closed system; $\tau=1/4K^2N_0t$ - dimensionless time.

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Following the first method of the introduction of coordinate functions, we seek the solution of boundary-value problem (4.33)-(4.35) in the form

$$U_n(x_0,\tau) = \sum_{k=1}^{n} C_k(\tau) \cos \frac{(2k-1)x_0}{2}, \qquad (4.36)$$

where $C_k(\tau)$- 'uncti' that are subject to further determination. With the weight factor in expression (4.22) $\rho(x)=1$ from (4.20) we will obtain system n of the ordinary differential equations:

$$\frac{dC_l(\tau)}{d\tau} = \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos^2 \frac{(2l-1)x}{2} dx\right]^{-1} \sum_{k=1}^{n} C_k(\tau) \left[\alpha(2k-1) \times \right.$$

$$\times \int_{-\pi}^{\pi} \sin x \sin \frac{(2k-1)x}{2} \cos \frac{(2l-1)x}{2} dx -$$

$$\left. - \frac{(2k-1)^2}{2} \int_{-\pi}^{\pi} \cos \frac{(2k-1)x}{2} \cos \frac{(2l-1)x}{2} dx \right], \quad l = 1, 2, \ldots, n. \qquad (4.37)$$

Since in this example the coefficients of equation (4.33) do not depend on time τ, then system (4.37) is the system of homogeneous equations with the constant coefficients. The matrix/die of coefficients with unknown functions $C_k(\tau)$ takes the form

$$a=\left\|\begin{array}{cccccccccc}
\frac{\alpha}{4}-1 & \frac{3\alpha}{4} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
-\frac{\alpha}{4} & -\frac{9}{4} & \frac{5\alpha}{4} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
0 & -\frac{3\alpha}{4} & -\frac{25}{4} & \frac{7\alpha}{4} & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & \frac{\alpha(-2k+3)}{4} & -\frac{(2k-1)^2}{4} & \frac{\alpha(2k+1)}{4} & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\frac{(2n-1)^2}{4}
\end{array}\right\|$$

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The solution of the system of homogeneous equations (4.37) with the constant coefficients let us represent in the form of the linear combination of the particular solutions

$$C_l(\tau)=\sum_{k=1}^{n} c_k A_{lk} e^{\lambda_k \tau}, \qquad (4.38)$$

where λ_k — the eigenvalues of matrix/die a; A_k — the corresponding to them eigen vectors. Eigenvalues λ_k are the roots of the characteristic equation

$$\det(a-\lambda I)=0,$$

where I - unit matrix. Eigenvectors $\mathbf{A}_k$ are determined with an accuracy to factors c_k, which are introduced into expression (4.38). For the certainty let us assume that elements/cells A_{kl} of eigenvectors are equal to the subdeterminants of matrix/die a, to equivalent components a_{kl}. After the calculation of eigenvectors $\mathbf{A}_k$ coefficients c_k are found from the system of the algebraic equations:

$$\sum_{k=1}^{n} c_k A_{lk} = C_l(0),\quad l = 1, 2, \ldots, n,$$

where the initial conditions

$$C_l(0) = (-1)^{l+1} \frac{4}{(2l-1)\pi}$$

are determined by op to formula (4.21).

Lowering intermediate linings/calculations, let us write out two first approximations:

the first approximation

$$U_1(x_0, \tau) = \frac{4}{\pi} e^{\lambda_1^{(1)}\tau} \cos\frac{x_0}{2}, \qquad (4.39)$$

where

$$\lambda_1^{(1)} = -\frac{1-\varepsilon}{4};$$

second approximation/approach

$$U_2(x_0, \tau) = \left[A e^{\lambda_1^{(2)}\tau} + \left(\frac{4}{\pi} - A\right) e^{\lambda_2^{(2)}\tau}\right] \cos\frac{x_0}{2} -$$

$$-\left[\frac{9 + 4\lambda_2^{(2)} - 3\varepsilon}{3\pi\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right)} e^{\lambda_1^{(2)}\tau} + \right.$$

$$\left. + \frac{\varepsilon}{9 + 4\lambda_2^{(2)}} \left(\frac{4}{\pi} - A\right) e^{\lambda_2^{(2)}\tau}\right] \cos\frac{3x_0}{2}, \qquad (4.40)$$

where

$$A=\frac{(9+4\lambda_2^{(2)}-3\alpha)\,(9+4\lambda_1^{(2)})}{3\pi\alpha\,(\lambda_2^{(2)}-\lambda_1^{(1)})};$$

$$\lambda_{1,2}^{(2)}=-\frac{10-\alpha}{8}\pm\frac{1}{8}\sqrt{64+16\alpha-11\alpha^2}.$$

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Superscripts in eigenvalues $\lambda_k^{(n)}$ indicate number of approximation/approach.

On the obtained relationships/ratios on Fig. 4.1 are constructed the graphs of the probability of the absence of phase jump-overs for the time τ during zero initial disagreement/mismatch (x_0=0). In the figure are constructed three first approximations (numeral indicates the number of approximation/approach). Solid li e constructed the curves, which correspond $\rho(x)=1$, to dash $-\rho(x)=w_{ст}(x)$. As can be seen from (4.39), in the signal-to-noise ratios $\alpha \geqslant 1$ first approximation gives physically inaccurate results - function $U(\tau)$ begins to increase in the time. However, up to the values α=0.5-0.7 the accuracy of formula (4.39) is completely sufficient for the practical calculations. Second approximation/approach (4.40) with $\alpha > \frac{1}{\pi}(1+2\sqrt{3}) \approx 3.24$ has complex eigenvalues, which also contradicts physical sense, since boundary-value problems for the one-dimensional equations of Fokker - Planck have the real spectrum, which lies at the negative region. Therefore with the large ones α it is necessary to compute higher approximations/approaches.

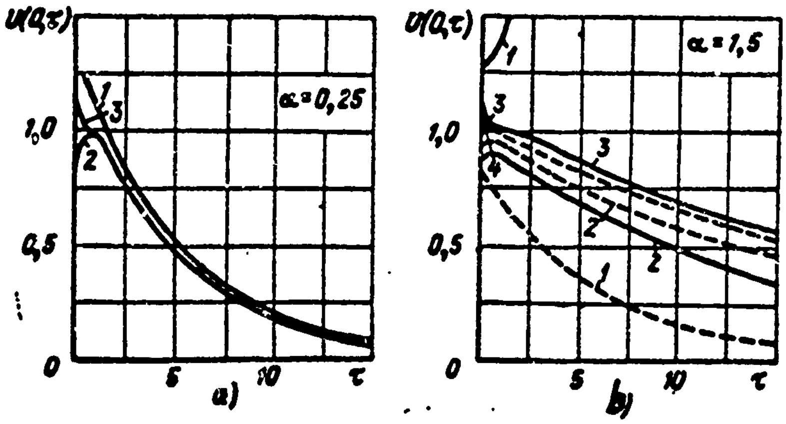

Fig. 4.1. The first approximations of the method of Bubnov - Galerkin.

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Following the second method of the introduction of coordinate functions, the solution of boundary-value problem (4.33)-(4.35) we seek in the form

$$U_n(x_0, \tau) = \frac{1}{\sqrt{w_{ст}(x_0)}} \sum_{k=1}^{n} C_k(\tau) \cos \frac{(2k-1)\, x_0}{2}. \quad (4.41)$$

Due to the symmetry of coefficient $A(x_0) = -\alpha \sin x_0$ and boundary conditions (4.35) in expansion (4.41) are left only cosinusoidal terms.

Steady-state solution (4.27) of the equation of Fokker - Planck in this example takes the form

$$w_{ст}(x) = e^{\alpha \cos x}.$$

System of equations (4.31) appears as follows:

$$\frac{dC_l(\tau)}{d\tau}=-\left[\frac{2l-1}{2}\right]^2 C_l(\tau)-\frac{\alpha}{\pi}\sum_{k=1}^{n} C_k(\tau)\int_0^{\pi}\left[\frac{\alpha}{2}\sin^2 x-\right.$$
$$\left.-\cos x\right]\cos\frac{(2k-1)x}{2}\cos\frac{(2l-1)x}{2}\,dx. \qquad (4.42)$$

On the basis (4.21) we will obtain initial conditions for unknown functions $C_k(\tau)$:

$$C_k(0)=\frac{2}{\pi}\int_0^{\pi} e^{\frac{\alpha}{2}\cos x}\cos\frac{(2k-1)x}{2}\,dx.$$

From (4.42) it follows that the eigenvalues with the large numbers are approximately equal to $\lambda_k\approx-(2k-1)^2/4$. The eigenvalues of first two approximations/approaches are computed in the quadratures

$$\lambda_1^{(1)}=-\frac{\alpha^2-2\alpha+2}{8}, \qquad (4.43)$$
$$\lambda_{1,2}^{(2)}=-\frac{\alpha^2-\alpha+10}{8}\pm\frac{1}{8}\sqrt{\frac{1}{4}\alpha^4+2\alpha^3+5\alpha^2+16\alpha+64}.$$

From comparison (4.43) with (4.39) and (4.40) it is evident that with small ones α the first two eigenvalues, found with two methods, virtually coincide. This is explained by the fact that weight factor $w_{\text{ст}}(x)$ with the decrease α approaches constant value. The advantages of the second method are revealed/detected with the large ones α. From (4.43) it follows that the eigenvalues, fo[illegible]d with the second method, are negative at any values of signal-to-noise ratio α.

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Fig. 4.1b shows first four approximations/approaches (4.41) with α=1.5. From the figure one can see that with selection of coordinate functions $\varphi_k(x_0)$ by the second method the convergence of the method of Bubnov - Galerkin is improved.

In Table 4.2 it is shown, as are stabilized eigenvalues $\lambda_k^{(n)}$ with an increase in the number of approximation/approach n (is used the second method, α=1.5).

In conclusion let us note that the expansion in terms of the system of coordinate functions (4.29) is expedient at the values α>1-1.5, when substantially deteriorates the convergence of expansions in terms of the set of functions (4.23) without taking into account weight factor (4.28). With small ones α both the methods examined give virtually identical results; however, the first method of the introduction of coordinate functions is simpler.

Example 2. In work [46] the method of Bubnov - Galerkin was used for solving the equation of Fokker - Planck, the probability of disruption/separation was determined from formula (2.77). As the system of coordinate functions were used trigonometric functions (4.23) with the weight factor $\rho(x)$=1.

Table 4.2.

(1) Метод решения	n	$\lambda_1^{(n)}$	$\lambda_2^{(n)}$	$\lambda_3^{(n)}$	$\lambda_4^{(n)}$	$\lambda_5^{(n)}$
(2) Бубнова—Галеркина	1	−0,156250	—	—	—	—
	2	−0,049136	−2,638364	—	—	—
	3	−0,042033	−2,609271	−6,567446	—	—
	4	−0,041804	−2,606381	−6,545740	−12,55306	—
	5	−0,041892	−2,606362	−6,543855	−12,53927	−20,51987
	6	−0,041892	−2,606361	−6,543848	−12,53789	−20,53632
(3) Асимптотический		−0,490972	−2,523413	−6,528220	−12,52967	−20,53028

Key: (1). Method of solution. (2). Bubnov - galerkina. (3). Asymptotic.

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With $A(x_0)=-A(-x_0)$, $B(x_0)=B_0$ and $x_0=0$ is obtained the following expression for probability of the disruption/separation:

$$P(0,t)\approx 1-\left(\frac{4}{\pi}-\frac{6A_1-A_3}{3\pi^2B_0^2}\right)\exp\left\{-\left[\left(\frac{\pi}{2\gamma}\right)^2\frac{B_0^2}{2}-\right.\right.$$

$$\left.\left.-\frac{\pi}{4\gamma^2}A_1\right]t\right\}-\left(\frac{5A_1}{2\pi^2B_0^2}-\frac{4}{3\pi}\right)\exp\left\{-\left[\left(\frac{3\pi}{2\gamma}\right)^2\frac{B_0^2}{2}-\right.\right.$$

$$\left.\left.-\frac{3\pi}{4\gamma}A_3\right]t\right\}-\left(\frac{2}{5\pi}-\frac{3A_1-2A_3}{6\pi^2B_0^2}\right)\exp\left[-\left(\frac{5\pi}{2\gamma}\right)^2\times\right.$$

$$\left.\times\frac{B_0^2}{2}t\right], \qquad (4.44)$$

where

$$A_k=\int_{-\gamma}^{\gamma}A(x_0)\sin\frac{k\pi x_0}{\gamma}\,dx_0.$$

Expression (4.44) is arithmetic mean of the second and third approximations/approaches, moreover the members of order B_T^4 and above are rejected/thrown. It is clear that the probability of disrupting/separating the tracking must be the function, which not decreases in the time. Expression (4.44) satisfies this condition only in such a case, when the indices of all exponential curves are negative. Usually $\Lambda_1>\Lambda_2>\Lambda_3>\ldots$ therefore solution (4.44) makes sense with satisfaction of the condition

$$\frac{\pi B_0^2}{2}>\Lambda_1, \qquad (4.45)$$

i.e. on the sufficiently high noise level.

4.3. Asymptotic method.

In the previous paragraph the solution of boundary-value problem for the probability of disruption/separation is represented in the form of series/row along the system of improper functions. The method of Bubnov - Galerkin makes it possible to efficiently find out the dominant terms of series/row, i.e., members, who correspond to small in the absolute value eigenvalues λ_k. The determination of the highest approximations/approaches is connected with the considerable computational difficulties. For the leading terms of series/row can be used asymptotic expansions of eigen functions [13].

The determination of the probability of the absence of disruption/separation $U(x_0, t)$ for time t is reduced to the solution of boundary-value problem with uniform boundary conditions (4.11)—(4.13). If the coefficients of equation (4.11) do not depend on time, then as a result of separation of variables (4.14) for functions T(t) and $X(x_0)$ we will obtain equations (4.15). Since eigenvalues $\lambda<0$, then for the convenience subsequently let us assume $\lambda=-\mu^2$. From (4.15) we find

$$T(t)=Ce^{-\mu^2 t}, \tag{4.46}$$

$$\frac{1}{2}B(x_0)X''+A(x_0)X'+\mu^2 X=0. \tag{4.47}$$

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Let us consider the possibility of solving equation (4.47) at the high values of the parameter μ^2. By the replacement of the variable/alternating

$$X'(x_0)=X(x_0)Z(x_0). \tag{4.48}$$

equation (4.47) is reduced to the equation of Riccati:

$$Z^2+Z'+\frac{2A}{B}Z+\frac{2}{B}\mu^2=0. \tag{4.49}$$

Let us represent solution of $Z(x_0)$ in the form of the asymptotic series/row

$$Z(x_0)\sim\mu\varphi_0(x_0)+\varphi_1(x_0)+\frac{\varphi_2(x_0)}{\mu}+\dots \tag{4.50}$$

The mino. totals of asymptotic series/row approximate well function $Z(x_0)$ at the high values μ. For finding of unknown functions φ_0, φ_1, φ_2 we differentiate (4.50) and will substitute result in equation (4.49), leaving only terms with the degrees μ^2, μ and μ^0:

$$\mu^2\varphi_0^2+\varphi_1^2+2\varphi_0\varphi_1\mu+2\varphi_0\varphi_2+\mu\varphi'_0+\varphi'_1+\frac{2A}{B}\varphi_0\mu+$$
$$+\frac{2A}{B}\varphi_1+\frac{2}{B}\mu^2+\ldots=0.$$

Gathering terms with the identical degrees μ, we will obtain system of equations relative to functions φ_0, φ_1, φ_2. As a result of its solution we find

$$\varphi_0(x_0)=\pm i\sqrt{\frac{2}{B(x_0)}},\quad \varphi_1(x_0)=\frac{B'(x_0)-4A(x_0)}{4B(x_0)},$$
$$\varphi_2(x_0)=\mp i\frac{\sqrt{B(x_0)}}{2\sqrt{2}B^2(x_0)}\left\{A^2(x_0)+A'(x_0)B(x_0)-\right.$$
$$\left.-A(x_0)B'(x_0)-\frac{B(x_0)B''(x_0)}{4}+\frac{3}{16}[B'(x_0)]^2\right\}. \tag{4.51}$$

Since $X'=XZ$, then

$$X(x_0)=\exp\left\{\int_c^{x_0}Z(\zeta)\,d\zeta\right\}.$$

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Consequently, two asymptotic solutions of equation (4.47) take the form:

$$X_{\pm}(x_0)=\frac{1}{\sqrt[4]{B(x_0)}\sqrt{w_{st}(x_0)}}\exp\left\{\pm i\left[\mu g(x_0)-\frac{1}{\mu}h(x_0)\right]\right\}, \tag{4.52}$$

where

$$g(x)=\int_c^x\sqrt{\frac{2}{B(\zeta)}}d\zeta;$$

$$h(x)=\int_c^x|\gamma_c(\zeta)|d\zeta;$$

$w_{st}(x)$ — steady-state solution (4.27) of the equation of Fokker - Planck (4.26) with the reflecting boundaries at points γ_1, γ_2.

Since the constants of integration can be attributed to the unknown thus far factor C in expression (4.46), then in solution (4.52) for the certainty let us place lower integration limit c equal to zero.

Let us represent the unknown solution of boundary-value problem in the form of the linear combination of particular solutions (4.52)

$$X(x_0)=C_1X_+(x_0)+C_2X_-(x_0) \tag{4.53}$$

and let us require so that at the points γ_1, γ_2 asymptotic solution (4.53) would be converted into zero according to boundary conditions (4.13):

$$C_1X_+(\gamma_1)+C_2X_-(\gamma_1)=0,$$
$$C_1X_+(\gamma_2)+C_2X_-(\gamma_2)=0.$$

Constants C_1 and C_2 are determined as a result of solving the

obtained uniform system of equations. Since solution must be nontrivial, then the determinant of system must become zero. From this condition we obtain equation for the eigenvalues

$$\sin\left\{\mu\left[g(\gamma_2)-g(\gamma_1)\right]-\frac{1}{\mu}\left[h(\gamma_2)-h(\gamma_1)\right]\right\}=0.$$

Hence we obtain the quadratic equation relatively μ, of two solutions of which makes sense only one:

$$\mu_k=\frac{k\pi+\sqrt{k^2\pi^2+4\left[g(\gamma_2)-g(\gamma_1)\right]\left[h(\gamma_2)-h(\gamma_1)\right]}}{2\left[g(\gamma_2)-g(\gamma_1)\right]}. \quad (4.54)$$

Second solution of quadratic equation gives the values μ, close to zero, with which asymptotic solution (4.50) is not correct. With large k formula (4.54) is simplified:

$$\mu_k\approx\frac{k\pi}{g(\gamma_2)-g(\gamma_1)}+\frac{h(\gamma_2)-h(\gamma_1)}{k\pi}. \quad (4.55)$$

Let us now find the asymptotic representation of eigenfunctions $X_k(x_0)$. In expression (4.53) one of the constants is chosen arbitrarily. Let $C_1=X_-(\gamma_2)$.

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Then for the satisfaction to boundary conditions (4.13) $C_2=-X_+(\gamma_2)$. With an accuracy to constant the eigenfunction

$$X_k(x_0)=X_-(\gamma_2)X_+(x_0)-X_+(\gamma_2)X_-(x_0). \quad (4.56)$$

The solution of boundary-value problem (4.11)-(4.13) is

written/recorded in the form

$$U(x_0, t) = \sum_{k=1}^{\infty} a_k e^{-\mu_k^2 t} X_k(x_0).$$

Coefficients a_k are determined from the expansion of initial condition (4.12) $U(x_0, 0)=1$ in the series/row in terms of system $X_k(x_0)$:

$$1 = \sum_{k=1}^{\infty} a_k X_k(x_0). \qquad (4.57)$$

For determining the coefficients a_k from (4.57) it is necessary to find set of functions Y_k, orthogonal to the eigenfunctions of equation (4.47). From the theory of linear differential operators it is known that the eigenfunctions of the adjoint equations are orthogonal. The equation, conjugated/combined to (4.47), is the equation

$$\frac{1}{2}(BY)'' - (AY)' + \mu^2 Y = 0. \qquad (4.58)$$

Using a property of orthogonality, let us multiply both parts of equality (4.57) to eigenfunctions Y_k of equation (4.58) and will integrate from γ_1 to γ_2. As a result let us find the coefficients

$$a_k = \left[\int_{\gamma_1}^{\gamma_2} X_k(x) Y_k(x)\,dx\right]^{-1} \int_{\gamma_1}^{\gamma_2} Y_k(x)\,dx. \qquad (4.59)$$

The asymptotic solutions of equation (4.58) are located by the same method such as is obtained the solution of equation (4.47):

$$Y_{\pm}(x) = \frac{\sqrt{w_{st}(x)}}{\sqrt[4]{B(x)}} \exp\left\{\pm i\left[\mu g(x) - \frac{1}{\mu} h(x)\right]\right\}. \qquad (4.60)$$

The eigenvalues, which correspond to eigenfunctions $Y_k(x)$, coincide with the spectrum of functions $X_k(x)$. From (4.52) and (4.60) it follows that the asymptotic solutions of straight line and conjugated/combined of equations are connected with the relationship/ratio

$$Y_k(x) = w_{0v}(x) X_{k'}(x). \qquad (4.61)$$

Substituting (4.61) in (4.59), we will obtain the resultant expression for the coefficients

$$a_k = \left[\int_{\gamma_1}^{\gamma_2} w_{0x}(x) X_k^2(x)\, dx \right]^{-1} \int_{\gamma_1}^{\gamma_2} w_{0x}(x) X_k(x)\, dx. \qquad (4.62)$$

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Thus, is obtained the asymptotic solution of boundary-value problem (4.11)-(4.13), valid with large ones

$$U(x_0, t) = \sum_k \frac{\int_{\gamma_1}^{\gamma_2} w_{0z}(x) X_k(x)\, dx}{\int_{\gamma_1}^{\gamma_2} w_{0z}(x) X_k^2(x)\, dx} e^{-\mu_k^2 t} X_k(x_0). \qquad (4.63)$$

Substituting in (4.63) eigenfunctions (4.56) and taking into account (4.52), we convert asymptotic solution to the form

$$U(x_0, t) = \sum_k d_k e^{-\mu_k^2 t} \frac{\sin\left\{\mu_k [g(x_0) - g(\gamma_1)] - \frac{1}{\mu_k}[h(x_0) - h(\gamma_1)]\right\}}{\sqrt[4]{B(x_0)}\sqrt{w_{0z}(x_0)}}, \qquad (4.64)$$

where

$$d_k = \frac{\int_{\gamma_1}^{\gamma_2} \sqrt{w_{0z}(x)}\, B^{-1/4}(x) \sin\left\{\mu_k [g(x) - g(\gamma_1)] - \frac{1}{\mu_k}[h(x) - h(\gamma_1)]\right\} dx}{\int_{\gamma_1}^{\gamma_2} B^{-1/2}(x) \sin^2\left\{\mu_k [g(x) - g(\gamma_1)] - \frac{1}{\mu_k}[h(x) - h(\gamma_1)]\right\} dx}.$$

It is possible to show that if $\gamma_2 = -\gamma_1 = \gamma$, $A(x) = -A(-x)$ and $B(x) = B(-x)$, then

$$U(x_0, t) = \sum_{k} \eta_k e^{-\mu_k^2 t} \frac{\cos\left(\mu_k g(x_0) - \frac{1}{\mu_k} h(x_0)\right)}{\sqrt[4]{B(x_0)}\sqrt{w_{0\tau}(x_0)}}, \quad (4.65)$$

where

$$\eta_k = (-1)^{(k+1)/2} \frac{\int\limits_0^{\gamma} \frac{\sqrt{w_{0\tau}(x)}}{\sqrt[4]{B(x)}} \cos\left(\mu_k g(x) - \frac{1}{\mu_k} h(x)\right) dx}{\int\limits_0^{\gamma} \frac{1}{\sqrt{B(x)}} \cos^2\left(\mu_k g(x) - \frac{1}{\mu_k} h(x)\right) dx},$$

$$k = 1, 3, 5, \ldots$$

The obtained asymptotic solutions are valid with eigenvalues $\lambda_k = -\mu_k^2$. large by the absolute value Small eigenvalues can be found with Bubnov- Galerkin method (see S 4.2).

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While the first approximations, which correspond to eigenvalues small in the absolute value, found with Bubnov-Galerkin method, describe solution with large t, asymptotic approximations/approaches they make more precise solution with small t.

In conclusion let us consider an example of system FAPCh (4.33), for which the first eigenvalues are found in S 4.2 with Bubnov-Galerkin method. Let us recall that in the analyzed case of A(x) =$-\alpha$ sin x, B(x)=2, $\gamma=\pi$. The probability not of disruption/separation, calculated according to formula (4.65), it

takes the form

$$U(x_0, \tau) = e^{-\frac{\alpha}{2}\cos x_0} \sum_{l=1}^{\infty} c_l e^{-\mu_l^2 \tau} \cos\left(\mu_l x_0 - \frac{1}{\mu_l} h(x_0)\right), \quad (4.66)$$

where

$$c_l = (-1)^l \frac{\int_0^\pi e^{\frac{\alpha}{2}\cos x} \cos\left(\mu_l x - \frac{1}{\mu_l} h(x)\right) dx}{\int_0^\pi \cos^2\left(\mu_l x - \frac{1}{\mu_l} h(x)\right) dx};$$

$$h(x) = \frac{\alpha^2}{16} x - \frac{\alpha^2}{32} \sin 2x - \frac{\alpha}{4} \sin x,$$

$\mu_l \approx \frac{2l-1}{2} + \frac{\alpha^2}{8(2l-1)}, \quad \tau = \frac{K^2 N_0}{4} t$ — dimensionless time.

For the comparison of asymptotic method with Bubnov-Galerkin method Table 4.2 gives the sequence of eigenvalues $\lambda_l = -\mu_l^2$ with α=1.5. From the table it is evident that already the second eigenvalues, found with both methods, are close to each other. Therefore the first two terms of the expansion of the solution of boundary-value problem should be found out by Bubnov-Galerkin method, and with $l \geq 3$ used asymptotic approximation/approach (4.66). To use only a first approximation according to Bubnov-Galerkin method is impossible, since in this case we obtain eigenvalue $\lambda_1^{(1)}$ with the large error.

4.4. Method of compensating sources.

During the solution of the problems about the first reaching/achievement by process of x(t) of boundaries γ_1, γ_2 the equation of Fokker-Planck is assigned only in the limited region Ω. The exact solution of boundary-value problem for this equation usually causes large mathematical difficulties and very rarely it can be found explicitly. At the same time for some tasks without the special labor/work it is possible to find the solution of the equation of Fokker-Planck, spread to entire infinite phase space.

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Under the initial conditions of form (2.41) this solution is called the fundamental solution of the problem of Cauchy (see S 2.3). In particular, if the analyzed process x(t) can be represented as the result of the passage of white noise through the linear device/equipment with the rational-fractional transfer function, then the fundamental solution of the problem of Cauchy for the equation of Fokker-Planck will be the normal law of the probability distribution whose parameters are comparatively easily located by the methods of correlation theory. On the basis of known fundamental solution it is possible to design the solution of boundary-value problem [72, 73, 81, 84].

Method of compensation. Let us consider preliminarily the

disruption/separation of tracking in the system, linear in the limits of the aperture of discriminator:

$$\left.\begin{array}{l}F(x)=Sx,\\ N_0(x)=N_0=\text{const}\end{array}\right\}\text{ with } \gamma_1<x<\gamma_2.$$

By disruption/separation of tracking is understood the first output of coordinate x beyond the boundaries γ_1, γ_2. Let the following error x(t) be the component of n-dimensional Markov process $\mathbf{x}(t)=\{x_1=x(t), x_2, \ldots, x_n\}$. The state of vector x(t) at the moment of the beginning of observation t=0 let us designate $\mathbf{x}_0=\{x_{01}, x_{02}, \ldots, x_{0n}\}$.

The equation of Fokker-Planck for the probability density of transition w(x, t; x_0) in the general case takes form (2.27). The unknown probability of disruption/separation is computed from the formula

$$P(t)=1-\int_{\Omega} w(\mathbf{x}, t; \mathbf{x}_0)\,d\mathbf{x}, \qquad (4.67)$$

where Ω - n-dimensional phase space, limited on coordinate x_1 by the absorbing boundaries of $x_1=\gamma_1$, $x_1=\gamma_2$; w(x, t; x_0) - the solution of the equation of Fokker-Planck (2.27), supplemented by the boundary conditions

$$w(\mathbf{x}, 0; \mathbf{x}_0)=\delta(\mathbf{x}-\mathbf{x}_0), \qquad (4.68)$$

$$w(\mathbf{x}, t; \mathbf{x}_0)\big|_{\mathbf{x}\in\tilde{G}}=0. \qquad (4.69)$$

Here $\tilde{G}$ - regular part of boundary G of phase space Ω.

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Let us continue the linear characteristic of discriminator F(x) =Sx to entire region $-\infty<x<\infty$ and we will temporarily consider that there are no absorbing boundaries. Then the fundamental solution of the problem of Cauchy satisfies equation (2.27) and initial condition (4.68) and is n-dimensional normal law (2.44). Let us designate this solution of $w_0(x, t; x_0)$. Function $w_0(x, t; x_0)$ yet is not the solution of boundary-value problem (2.27), (4.68)-(4.69), since on the regular part of $\tilde{G}$ of boundary fundamental solution does not become zero. For the compensation for probability density on $\tilde{G}$ let us place beyond the limits of region Ω the series/row of the further sources of density so, in order to at the initial moment t=0

$$w(x, 0; x_0)=\delta(x-x_0)-\sum_{i=1}^{N} a_i\delta(x-x_i), \qquad (4.70)$$

where a_i— unknown thus far coefficients of the intensities of the further sources, arranged/located at zero time at points x_i.

Since poles x_i of auxiliary sources are arranged/located beyond the limits of the absorbing boundaries, latter/last recording does not contradict condition (4.68) of boundary-value problem. In view of the linearity of the equation of Fokker-Planck his solution taking into account (4.70) can be represented as the superposition of the fundamental solutions

$$w(\mathrm{x}, t; \mathrm{x}_0) = w_0(\mathrm{x}, t; \mathrm{x}_0) - \sum_{i=1}^{N} a_i w_0(\mathrm{x}, t; \mathrm{x}_i). \quad (4.71)$$

Let us select coefficients a_i in such a way that the resulting probability density on the regular part of the boundary

$$\left[w_0(\mathrm{x}, t; \mathrm{x}_0) - \sum_{i=1}^{N} w_0(\mathrm{x}, t; \mathrm{x}_i)\right]_{\mathrm{x}\in\tilde{G}}$$

would vanish with an increase in number N of commutating poles. This makes it possible to consider combination (4.71) approximate solution of initial boundary-value problem. Increasing a number of commutating poles, we obtain increasingly more degrees of freedom in the selection of coefficients a_i in order to approach (4.71) the exact solution.

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In the limit with $N\to\infty$ it is possible to obtain the exact solution of boundary-value problem in the form

$$w(\mathrm{x}, t; \mathrm{x}_0) = w_0(\mathrm{x}, t; \mathrm{x}_0) - \int_{\bar{\Omega}} \alpha(\mathrm{z})\, w_0(\mathrm{x}, t; \mathrm{z})\, d\mathrm{z}, \quad (4.72)$$

where $\bar{\Omega}$ - entire n-dimensional space, with exception of region Ω.

Weight function $\alpha(z)$ is determined from boundary condition (4.69), which taking into account (4.72) takes the form

$$w_0(\mathrm{x}, t; \mathrm{x}_0)\Big|_{\mathrm{x}\in\tilde{G}} = \int_{\bar{\Omega}} \alpha(\mathrm{z})\, w_0(\mathrm{x}, t; \mathrm{z})\Big|_{\mathrm{x}\in\tilde{G}} d\mathrm{z}. \quad (4.73)$$

Thus, the solution of boundary-value problem succeeds in reducing to the composition of linear combination (4.72) and the solution of the auxiliary integral equation of Fredholm first kind (4.73). To some simple examples of the solution of boundary-value problems by the method examined it is possible to be introduced in [72].

Approximate approach. In connection with the fact that the exact solution of the equation of Fredholm (4.73) in the majority of the cases to find difficultly, let us pause at the approximate method of solving the boundary-value problems by the method of compensating sources [81].

After taking as the basis solution in the form of finite series (4.71), let us bound a quantity of further sources with a number of absorbing boundaries and will place reversing poles into the points of the mirror reflection of the basic pole x_0 relative to boundaries γ_1, γ_2:

$$\begin{aligned} \mathbf{x}_1 &= \{2\gamma_1 - x_{01}, -x_{02}, \ldots, -x_{0n}\}, \\ \mathbf{x}_2 &= \{2\gamma_2 - x_{01}, -x_{02}, \ldots, -x_{0n}\}. \end{aligned} \tag{4.74}$$

Taking into account that the phase space Ω is limited only on coordinate x_1, let us switch over to one-dimensional probability densities, after integrating both parts (4.71) with respect to

variable/alternating $x_2, \ldots, x_n$ in the infinite limits.

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As a result the probability of disruption/separation is equal to

$$P(t_n) = 1 - \int_{\gamma_1}^{\gamma_2} w(x, t_n; x_0)\,dx, \qquad (4.75)$$

where one-dimensional density $w(x, t_n; x_0)$ is approximately determined by the linear combination

$$w(x, t_n; x_0) \approx w_0(x, t_n; x_0) - -a_1 w_0(x, t_n; x_1) - a_2 w_0(x, t_n; x_2). \qquad (4.76)$$

Let us note that after integration boundary conditions (4.69) for the one-dimensional densities in the general case cease to be uniform:

$$w(\gamma_{1(2)}, t; x_0) \neq 0.$$

However, taking into account that the boundaries γ_1, γ_2 the region of tracking, as a rule, coincide with the points of the unstable equilibrium of system or close to them, natural to expect near the boundaries of the very low value of probability density. Therefore approximately let us assume

$$w(x, t; x_0)|_{x=\gamma_1} \approx w(x, t; x_0)|_{x=\gamma_2} \approx 0. \qquad (4.77)$$

In particular, relationship/ratio (4.77) becomes precise, if entire/all boundary is regular ($\tilde{G}$=G). This it has locally, for example, in the servo system of the second order with

proportional-integrating filter.

Usually do not succeed in finding the values of coefficients α_1 and α_2 which would satisfy condition (4.77) and in this case they were not the functions of time t or coordinate x. Approximate solution of boundary-value problem can be found, after determining coefficients α_1, α_2, so that boundary conditions (4.77) would be satisfied only on the average within the time of the observation

$$\int_0^{t_H}\left\{w_0(\gamma_{1(2)}, t; x_0)-\sum_{i=1}^{2} \alpha_i w_0(\gamma_{1(2)}, t; x_i)\right\} dt=0. \quad (4.78)$$

Solving the system of linear algebraic equations (4.78), we find

$$\alpha_1=\frac{\bar{w}_{01}\bar{w}_{22}-\bar{w}_{21}\bar{w}_{02}}{\bar{w}_{11}\bar{w}_{22}-\bar{w}_{21}\bar{w}_{12}}, \quad \alpha_2=\frac{\bar{w}_{11}\bar{w}_{02}-\bar{w}_{01}\bar{w}_{12}}{\bar{w}_{11}\bar{w}_{22}-\bar{w}_{12}\bar{w}_{21}}, \quad (4.79)$$

where

$$\bar{w}_{ij}=\int_0^{t_H} w_0(\gamma_j, t; x_i)\, dt$$

- average/mean for the time of observation t_H probability density on boundary γ_j, caused by source with pole x_i.

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Let us consider some examples of the analysis of the disruption/separation of tracking in the regulating circuits by the method of the compensating sources.

Example 1. Let us calculate the probability of disruption/separation in first-order system (see Fig. 3.5) on the assumption that $d\lambda/dt=0$ and $F(x)=Sx$ with $-\gamma<x<\gamma$. Stochastic equation of the system being investigated takes the form

$$\frac{dx}{dt}=-KSx-K\sqrt{N_0}\xi^0(t), \qquad (4.80)$$

and the corresponding to it equation of Fokker-Planck

$$\frac{\partial w(x,t;x_0)}{\partial t}=KS\frac{\partial}{\partial x}(xw)+\frac{K^2N_0}{4}\frac{\partial^2 w}{\partial x^2}. \qquad (4.81)$$

Let up to moment/torque t=0 of the inclusion of noise the following error in the system take value of $x(0)=x_0$. The transiency of task is exhibited in the fact that dispersion $\sigma^2_x(t)$ and mathematical expectation $m_x(t)$ of following error during the transient process depend on time.

For determining the probability of disruption/separation it is necessary to solve equation (4.81), supplemented by the boundary conditions

$$w(x,0;x_0)=\delta(x-x_0), \qquad (4.82)$$

$$w(-\gamma,t;x_0)=w(\gamma,t;x_0)=0. \qquad (4.83)$$

The fundamental solution of the problem of Cauchy for equation (4.81) with initial condition (4.82) is the one-dimensional normal law

$$w_0(x,t;x_0)=\frac{1}{\sqrt{2\pi}\,\sigma_x(t)}\exp\left\{-\frac{(x-x_0e^{-\alpha t})^2}{2\sigma_x^2(t)}\right\}, \qquad (4.84)$$

where

$$\sigma^2_x(t)=\frac{KN_0}{4S}(1-e^{-2\alpha t}), \quad \alpha=KS.$$

In accordance with (4.74) let us place the poles of the

compensating sources into points $x_1=-2\gamma-x_0$, $x_2=2\gamma-x_0$. As a result the compensating for probability densities will take the form

$$w_0(x,t;x_1)=\frac{1}{\sqrt{2\pi}\,\sigma_x(t)}\exp\left\{-\frac{[x+(2\gamma+x_0)\,e^{-\alpha t}]^2}{2\sigma_x^2(t)}\right\},$$

$$w_0(x,t;x_2)=\frac{1}{\sqrt{2\pi}\,\sigma_x(t)}\exp\left\{-\frac{[x-(2\gamma-x_0)\,e^{-\alpha t}]^2}{2\sigma_x^2(t)}\right\}. \quad (4.85)$$

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The obtained expressions make it possible to determine in formulas (4.79) the coefficients of intensity α_1 and α_2. For the determination of the probability of disruption/separation let us integrate expression (4.76) in accordance with (4.75), taking into account the concrete/specific/actual form of fundamental solutions (4.84)-(4.85). As a result we will obtain

$$P(t_n)\approx 1-\frac{1}{2}\Phi\left(\frac{\gamma-x_0e^{-\alpha t_n}}{\sqrt{2}\,\sigma_x(t_n)}\right)-\frac{1}{2}\Phi\left(\frac{\gamma+x_0e^{-\alpha t_n}}{\sqrt{2}\,\sigma_x(t_n)}\right)+$$

$$+\frac{\alpha_1}{2}\Phi\left(\frac{\gamma+(2\gamma+x_0)\,e^{-\alpha t_n}}{\sqrt{2}\,\sigma_x(t_n)}\right)+\frac{\alpha_1}{2}\Phi\left(\frac{\gamma-(2\gamma+x_0)\,e^{-\alpha t_n}}{\sqrt{2}\,\sigma_x(t_n)}\right)+$$

$$+\frac{\alpha_2}{2}\Phi\left(\frac{\gamma-(2\gamma-x_0)\,e^{-\alpha t_n}}{\sqrt{2}\,\sigma_x(t_n)}\right)+\frac{\alpha_2}{2}\Phi\left(\frac{\gamma+(2\gamma-x_0)\,e^{-\alpha t_n}}{\sqrt{2}\,\sigma_x(t_n)}\right). \quad (4.86)$$

Some results of the numerical calculations, carried out according to formula (4.86), are given in Fig. 4.2, where are accepted the following designations: $Y=KN_0/S\gamma^2$, $X=x_0/\gamma$, $\tau=KSt_n$.

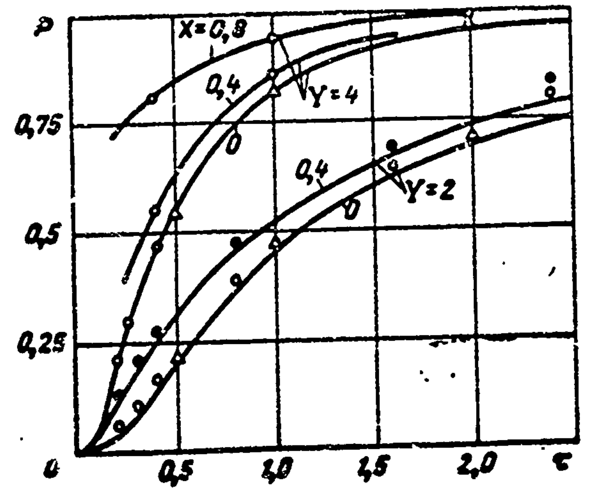

Fig. 4.2. The probability of disrupting/separating the tracking in first-order system: ——— - method of compensation; ● O - solution of boundary-value problem on AVM; Δ - results of work [65].

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On the accuracy of the solution of boundary-value problem it is possible to judge by the comparison of the obtained results with the results of other methods. In particular, the same initial equation of Fokker-Planck (4.81) with boundary conditions (4.82)-(4.83) was solved in the analog computer (see § 6.3) and, furthermore, with $x_*=0$ the solution of problem is compared with the more accurate results, found in [65]. From comparison it is evident that the method of the compensating sources for first-order system can give fair results during the calculation of the probability of disruption/separation up

to values of $P \lesssim 0.8$.

Example 2. is complicated the previous example, after supplementing into the feedback loop of servo system proportional-integrating filter, so as to the resulting gear ratio/transmission factor would become equal to $K(p)=K(1+pT_1)/p(1+pT)$. Stochastic equation. which describes the behavior of the analyzed system, takes form (2.18), and the two-dimensional equation of F[illegible]r-Planck - (2.38). The boundary conditions of the decided task [illegible]ain form (4.68)-(4.69). In this case the regular part $\tilde{G}$ of the boundary of the region of tracking on the phase plane consists of the lines

$$x=\pm\gamma, \text{ if } T_1>0,$$

or

$$x=\begin{cases}-\gamma & \text{при}^{(1)}\ \dot{x}>0,\\ \gamma & \text{при}^{(1)}\ \dot{x}<0,\end{cases}$$

Key: (1). with.

if $T_1=0$.

In the first case condition (4.77) becomes precise.

Following the general/common/total procedure of calculation of the probability of disruption/separation, let us determine one-dimensional area of transitional probability in the form (4.76). The functions, which compose linear combination (4.76), are the one-dimensional probability densities of the Gaussian process x(t):

$$w_0(x, t; x_0) = \frac{1}{\sqrt{2\pi}\,\sigma_x(t)} \exp\left\{-\frac{[x - m_x(x_0, t)]^2}{2\sigma_x^2(t)}\right\}. \quad (4.87)$$

Mathematical expectation and dispersion, entering distribution (4.87), are found directly from stochastic equation (2.18) of linear system, for example, with the help of its averaging and conversion according to Laplace [8]. In this example

$$m_x(x_0, t) = x_0 \frac{a e^{-at} - b e^{-bt}}{a-b} + [\dot{x}_0 + (\beta + K_v n) x_0]\, \frac{e^{-bt} - e^{-at}}{a-b},$$

$$\sigma_x^2(t) = \frac{N_0 K^2}{2}\left\{\frac{K_v n^2 + \beta}{2K_v(\beta + K_v n)} + \frac{2(an-\beta)(bn-\beta)}{(a-b)^2(a+b)} \times\right.$$

$$\left. \times e^{-(a+b)t} - \frac{(an-\beta)^2}{2a(a-b)^2} e^{-2at} - \frac{(bn-\beta)^2}{2b(a-b)^2} e^{-2bt}\right\},$$

where

$$a, b = \frac{1}{2}\left[\beta + K_v n \pm \sqrt{(\beta + K_v n)^2 - 4\beta K_v}\right],$$

$$K_v = KS, \ \beta = \frac{1}{T}, \ n = \frac{T_1}{T}.$$

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Assuming that at the moment of the beginning of observation, which coincides with the inclusion/connection of noise, the state of process x(t) is determined by parameters x_0 and $\dot{x}_0$, in accordance with (4.74) let us take the coordinates of commutating poles $x_1 = -2\gamma - x_0$, $x_2 = 2\gamma - x_0$, $\dot{x}_1 = \dot{x}_2 = -\dot{x}_0$. Thus, all parameters, entering

one-dimensional functions (4.87), are determined and it is possible to switch over to the calculation of the coefficients of compensation α_1, α_2 in formulas (4.79). With the obtained coefficients of compensation the unknown probability of disruption/separation is determined from formula (4.75), which in this case takes the form

$$P(t_k) \approx 1 + \sum_{i=0}^{2} \frac{\alpha_i}{2}\left\{\Phi\left(\frac{\gamma - m_x(x_i, t_k)}{\sqrt{2}\,\sigma_x(t_k)}\right) + \Phi\left(\frac{\gamma + m_x(x_i, t_k)}{\sqrt{2}\,\sigma_x(t_k)}\right)\right\}, \tag{4.88}$$

where $\alpha_0 = -1$.

In particular, with KST=0.2, $Y = KN_0/S\gamma^2 = 10$ according to formula (4.88) were carried out the calculations whose results were represented in Fig. 4.3 in the form of graphs (solid lines). Dotted line there showed more precise dependences, found by the simulation of servo system on TsVM.

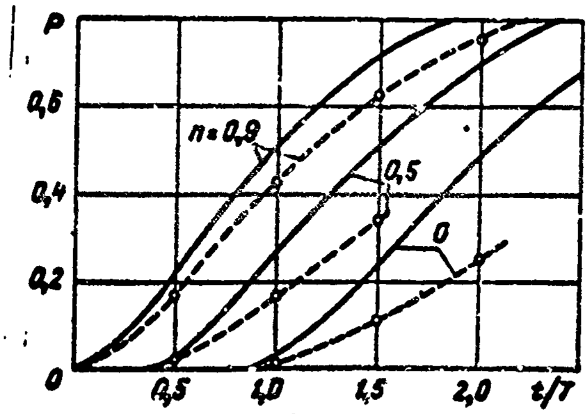

Fig. 4.3. Disruption/separation of tracking in the system with the proportional-integrating filter.

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Conclusions. As it follows from the examples examined, the method of compensation allows with an accuracy sufficient for the practice to determine the probability of disrupting/separating the tracking during the nonstationary systems of the work of regulating circuits. The very good accuracy of the determination of probability (10-20% with $P \leqslant 0.5-0.8$) occurs during the analysis of the systems of tracking with first-order filters.

From the comparison of Fig. 4.3 and 4.2 it is evident that during the analysis of the systems of the second order the method of the compensating sources gives greater error than during the analysis

of first-order systems. This is explained by the fact that with the determination of coefficients α_1 and α_2 and the multidimensional case besides the averaging of probability density on the boundary in the time is supplemented the averaging on space coordinates $x_2 \ldots, x_n$. Furthermore, in the multidimensional tasks boundary condition itself (4.77) for the one-dimensional densities becomes approximated, if the regular part of the boundary is only the part of the entire boundary of the region Ω ($\tilde{G} \neq G$). Therefore, as can be seen from Fig. 4.3, with n=0 is observed the greatest error in the solution, which at the level P=0.1 leads to the error in the determination of the permissible signal-to-noise ratio from the stress/voltage approximately/exemplarily to 20%.

4.5. Generalization of the method of the compensating sources to the nonlinear systems.

Formulation of the problem. In the previous paragraph were analyzed the servo systems with the linear discriminatory characteristic in the limits of entire aperture $\gamma_1<x<\gamma_2$. Much more frequent in the practice are encountered the systems whose discriminatory characteristics are substantially nonlinear. In the principle there is possibility [84] to conduct approximate analysis of such systems with the method of compensation, if we preliminarily linearize system in the limits of its aperture. However, known

linearization methods possess a comparatively large error. Therefore to more expediently analyze disruption/separation in the linearized systems on the basis of Poisson's law (see S 4.1), but not with the help of the method of compensation, since the latter, without removing errors in the linearization, requires sufficiently cumbersome calculations.

In this paragraph the method of compensation applies to nonlinear regulating circuits whose discriminatory characteristics can be approximated by the piecewise-linear dependences.

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Furthermore, it is assumed that the random effect $\xi(t)$ has the spectral density, which does not depend on disagreement/mismatch x. This approximately can be achieved/reached, for example, with the help of the averaging according to formula (3.20).

In S 2.3 it is shown that approximate solution of boundary-value problem for the equation of Fokker-Planck for the system with the characteristic of discriminator linear in the finite segment $\gamma_1<x<\gamma_2$ can be found in the form of the sum of fundamental solutions. Such solutions are determined for each linear section of the characteristic of discriminator and then "are joined" at the salient

points. For the join of solutions is used the continuity condition of the probability density and flow of probability [38, 42] upon transfer from one section of the characteristic of discriminator to another. Since approximate solution is sought in the form of the final sum of the fundamental solutions of the problem of Cauchy, for the join of solutions by analogy with the previous material is used the approximate criterion of the continuity of the probability density and flow on the average for the time of observation.

Analysis of first-order systems. Let us examine in more detail the methodology of the solution of boundary-value problem for the one-dimensional equation of Fokker-Planck in the case when the characteristic of discriminator consists of three linear sections (Fig. 4.4 - solid line). Let the boundary-value problem being subject to solution be determined by the one-dimensional equation of Fokker-Planck, by initial condition (4.[illegible]2) and boundary conditions

$$w(\gamma_{1(2)}, t; x_0) = 0. \tag{4.89}$$

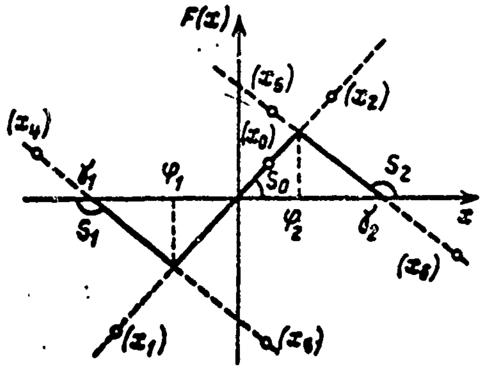

Fig. 4.4. Characteristics of discriminator.

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Let us find approximate solution of boundary-value problem for each linear section of the characteristic of discriminator F(x). In the section $\varphi_1 \leq x \leq \varphi_2$ solution let us represent in the form of the sum of three functions:

$$w(x, t; x_0) \approx w_0(x, t; x_0) + a_1 w_0(x, t; x_1) + + a_2 w_0(x, t; x_2), \tag{4.90}$$

being the fundamental solutions of the problem of Cauchy for the equation of Fokker-Planck, found on the assumption that characteristic F(x) is linear with slope/transconductance S_0 in the entire region $-\infty<x<\infty$ (Fig. 4.4). In this case solution $w_0(x, t; x_0)$ is obtained for the basic pole x_0, thanks to which is satisfied initial condition (4.82) of task, and solutions $w_0(x, t; x_1)$ and $w_0(x, t; x_2)$ are found for commutating poles x_1 and x_2, which lie beyond the limits of the salient points of characteristic. Solutions

$w_0(x, t; x_1)$, $w_0(x, t; x_2)$, undertaken with the weights α_1, α_2, are intended for the satisfaction of the conditions of join at points φ_1 and φ_2.

The solution of boundary-value problem in the section $\gamma_1 < x \leqslant \varphi_1$ let us represent also in the form of the sum of the fundamental solutions

$$w(x, t; x_0) \approx a_3 w_1(x, t; x_3) + a_4 w_1(x, t; x_4), \quad (4.91)$$

of those found on the assumption that the characteristic of discriminator is continued to infinity with slope/transconductance S_1, which occurred in the section being analyzed. The poles of solutions x_3, x_4 are chosen beyond the limits of section $\gamma_1 < x \leqslant \varphi_1$ in order not to break the initial condition of initial task.

Let us analogously register solution in the region $\varphi_2 \leqslant x < \gamma_2$ in the form

$$w(x, t; x_0) \approx a_5 w_2(x, t; x_5) + a_6 w_2(x, t; x_6), \quad (4.92)$$

where $w_2(x, t; x_5)$ and $w_2(x, t; x_6)$ - the fundamental solutions, obtained with $F(x) = S_2 x$, $-\infty < x < \infty$.

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As a result the solution of boundary-value problem takes the form:

$$w(x,t;x_0) \approx \begin{cases} a_3 w_1(x,t;x_3) + a_4 w_1(x,t;x_4) \\ \qquad \text{при (1)}\ \gamma_1 < x \leqslant \varphi_1, \\ w_0(x,t;x_0) + a_1 w_0(x,t;x_1) + \\ + a_2 w_0(x,t;x_2) \quad \text{при (1)}\ \varphi_1 \leqslant x \leqslant \varphi_2, \\ a_5 w_2(x,t;x_5) + a_6 w_2(x,t;x_6) \\ \qquad \text{при (1)}\ \varphi_2 \leqslant x < \gamma_2, \end{cases} \quad (4.93)$$

Key: (1). with.

where $w_i(x,\ t;\ x_j)$ — one-dimensional Gaussian probability densities (4.84); α_1-α_4 - coefficients of the intensities of further sources, which are subject to further determination.

Join of solutions. Strictly speaking, the solution of boundary-value problem must satisfy continuity conditions at the salient points of the characteristic of discriminator φ_i for the probability density

$$w(x,t;x_0)|_{x\to\varphi_i+0} = w(x,t;x_0)|_{x\to\varphi_i-0}, \quad i=1,2 \quad (4.94)$$

and for the flow of probability density

$$\Pi(x,t;x_0)|_{x\to\varphi_i+0} = \Pi(x,t;x_0)|_{x\to\varphi_i-0} \quad i=1,2. \quad (4.95)$$

Furthermore, solution must satisfy boundary conditions (4.89). During the recording of approximate solution in the form (4.93) for satisfaction of the enumerated conditions it is necessary to fit the appropriate coefficients α_1-α_4. However, for the majority of the practical tasks this cannot be done. Therefore it is expedient to

replace boundary conditions (4.89) and conditions of join (4.94)-(4.95) with those approximated, after requiring so that they would be implemented only on the average within the time of observation $t_н$.

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As a result for determining the coefficients of intensities we have a system of the linear algebraic equations:

$$
\left.
\begin{aligned}
&\int_0^{t_н} w_0(\varphi_1, t; x_0)\,dt + a_1 \int_0^{t_н} w_0(\varphi_1, t; x_1)\,dt + \\
&+ a_2 \int_0^{t_н} w_0(\varphi_1, t; x_2)\,dt = a_3 \int_0^{t_н} w_1(\varphi_1, t; x_3)\,dt + \\
&\qquad + a_4 \int_0^{t_н} w_1(\varphi_1, t; x_4)\,dt, \\
&\int_0^{t_н} w_0(\varphi_2, t; x_0)\,dt + a_1 \int_0^{t_н} w_0(\varphi_2, t; x_1)\,dt + \\
&+ a_2 \int_0^{t_н} w_0(\varphi_2, t; x_2)\,dt = a_3 \int_0^{t_н} w_1(\varphi_2, t; x_3)\,dt +
\end{aligned}
\right| \quad (4.96)
$$

$$\left.\begin{aligned}
&\qquad\qquad + \alpha_6 \int_0^{t_k} w_2(\varphi_2, t; x_6)\, dt, \\
&\int_0^{t_k} \Pi_0(\varphi_1, t; x_0)\, dt + \alpha_1 \int_0^{t_k} \Pi_0(\varphi_1, t; x_1)\, dt + \\
&+ \alpha_2 \int_0^{t_k} \Pi_0(\varphi_1, t; x_2)\, dt = \alpha_3 \int_0^{t_k} \Pi_1(\varphi_1, t; x_3)\, dt + \\
&\qquad\qquad + \alpha_4 \int_0^{t_k} \Pi_1(\varphi_1, t; x_4)\, dt, \\
&\int_0^{t_k} \Pi_0(\varphi_2, t; x_0)\, dt + \alpha_1 \int_0^{t_k} \Pi_0(\varphi_2, t; x_1)\, dt + \\
&+ \alpha_2 \int_0^{t_k} \Pi_0(\varphi_2, t; x_2)\, dt = \alpha_5 \int_0^{t_k} \Pi_2(\varphi_2, t; x_5)\, dt + \\
&\qquad\qquad + \alpha_6 \int_0^{t_k} \Pi_2(\varphi_2, t; x_6)\, dt, \\
&\alpha_3 \int_0^{t_k} w_1(\gamma_1, t; x_3)\, dt + \alpha_4 \int_0^{t_k} w_1(\gamma_1, t; x_4)\, dt = 0, \\
&\alpha_5 \int_0^{t_k} w_2(\gamma_2, t; x_5)\, dt + \alpha_6 \int_0^{t_k} w_2(\gamma_2, t; x_6)\, dt = 0,
\end{aligned}\right\}$$

where $\Pi_i(x, t; x_j)$ — flow of probability density, found under the same conditions as corresponding i-fundamental solution $w_i(x, t; x_j)$ wi h the pole at point x_j. Flow $\Pi_i(x, t; x_j)$ unambiguously is expressed as known fundamental solution $w_i(x, t; x_j)$. This connection/communication for each specific case follows from the comparison of the initial equation of Fokker-Planck with his divergent form of recording (2.36).

The solution of the system of algebraic equations (4.96) relative to coefficients α_1-α_4 does not cause fundamental difficulties; however, the process of calculating the definite integrals, entering the system, frequently proves to be very bulky, since it is necessary to apply numerical methods.

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Substantial to facilitate the process of calculating the coefficients α_1-α_4 it is possible after using TsVM; however, if ne essary all linings/calculations can be carried out by hand, for example, by graphic method. The a curacy of the calculation of definite integrals can be low, are sufficient to ensure it about 10%. Further increase in the accuracy does not lead to the refinement of the probability of disruption/separation due to errors in the method itself.

Let us pause at the selection of poles x_j of the further sources

of probability density. As has already been mentioned, their coordinates on x axis must be arranged/located beyond the limits of the corresponding working sections of the characteristic of discriminator, so that would not be broken initial condition (4.82). As showed numerical checking, solution (4.93) in this case is not susceptible to the position of further sources. By analogy with the material of the previous paragraph we consider that poles x_j are arranged/located mutually symmetrically relative to points φ_1, φ_2 and γ_1, γ_2. Thus, if basic pole has a coordinate x_0, then for the further ones let us assume

$$x_1=2\varphi_1-x_0,\; x_2=2\varphi_2-x_0,\; x_3=x_5=x_0, \qquad (4.97)$$
$$x_4=2\gamma_1-x_0,\; x_6=2\gamma_2-x_0.$$

During this selection of poles and in absence of external dynamic disturbance/perturbation ($d\lambda/dt=0$) the equation of Fokker-Planck in sections $x>\varphi_2$ and $x<\varphi_1$ becomes symmetrical relative to points γ_1 and γ_2. Because of the fact that poles x_3, x_4 and x_5, x_6 are mutually symmetrical relative to the same points, for satisfaction of the conditions for absorption (4.89) it suffices to assume $\alpha_4=-\alpha_3$, $\alpha_6=-\alpha_5$. In this case a number of unknown coefficients of intensities is decreased to four, which simplifies their determination. Finally, in the absence of dynamic disturbance/perturbation, to the symmetrical characteristic of discriminator $F(x)=-F(-x)$ and the initial disagreement/mismatch $x_0=0$ a number of independent coefficients is decreased up to two, since

$\alpha_1=\alpha_2$, $\alpha_3=\alpha_5=-\alpha_4=-\alpha_6$.

A special case. Let us consider the system of first-order (Fig. 3.5) automatic tracking with the integrator in the feedback loop $K(p)=K/p$.

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Let us determine the probability of disrupting/separating the tracking during the arbitrary determined input disturbance/perturbation $\lambda(t)$ and the random disturbance $\xi(t)$ in the form of white noise with a spectral density of N_0, which does not depend on following error x. Stochastic equation of the analyzed system takes the form

$$\frac{dx}{dt}=\frac{d\lambda}{dt}-K\left[F(x)+\sqrt{N_0}\xi^{0}(t)\right]. \tag{4.98}$$

Let at zero time the state of system be known accurate for $x(0)=x_0$. For calculating the probability of disruption/separation it is necessary to solve the boundary-value problem

$$\frac{\partial w(x,t;x_0)}{\partial t}+\frac{\partial}{\partial x}\left\{\left[\frac{d\lambda}{dt}-KF(x)\right]w\right\}=\frac{K^2N_0}{4}\frac{\partial^2 w}{\partial x^2},$$
$$w(x,0;x_0)=\delta(x-x_0),\ w(\gamma_{1(2)},t;x_0)=0 \tag{4.99}$$

and to fulfill integration for formula (4.75).

Let us assume that the characteristic of discriminator F(x) is approximated by the dependence, depicted in Fig. 4.4. In accordance

with the method of the compensating sources approximate solution of boundary-value problem (4.99) let us determine in the form (4.93), moreover commutating poles it is placed in accordance with (4.97). The probability densities, entering in (4.93), are described by the gaussian dependences

$$w_i(x, t; x_j) = \frac{1}{\sqrt{2\pi}\,\sigma_i(t)} \exp\left\{-\frac{[x - m_i(x_j, t)]^2}{2\sigma_i^2(t)}\right\},\ i = 0, 1, 2. \tag{4.100}$$

Taking into account that the initial equation of Fokker-Planck (4.99) in the divergent form takes the form

$$\frac{\partial w(x, t; x_0)}{\partial t} = -\frac{\partial \Pi(x, t; x_0)}{\partial x},$$

for the flow of probability density along x axis we will obtain depending on the section of the characteristic of discriminator the following expression:

$$\Pi_i(x, t; x_j) = w_i(x, t; x_j)\left[\frac{K^2 N_0 (x - m_i(x_j, t))}{4\sigma_i(t)} + \right.$$
$$\left. + \frac{d\lambda}{dt} - f_i(x)\right],$$

where

$$f_0(x) = \beta_0 x,\quad f_1(x) = \beta_1 (x - \gamma_1),\ f_2(x) = \beta_2 (x - \gamma_2),$$
$$\beta_i = KS_i,\quad i = 0, 1, 2,$$

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Dispersions $\sigma_i^2(t)$, the found with path averaging and the twofold conversion stochastic equation (4.98) according to Laplace [8], are equal to

$$\sigma_i^2(t) = \frac{N_0 K}{4S_i}\left[1 - \exp(-2\beta_i t)\right],$$

and mathematical expectations $m_i(x_j, t)$ are the solutions of the

differential equations

$$\frac{dm_i(x_j,t)}{dt}=\frac{d\lambda}{dt}-f_i(m_i), \quad m_i(x_j,0)=x_j,$$

which also follows from (4.98).

For the final solution of boundary-value problem (4.99) should be found the coefficients of intensities $\alpha_1-\alpha_4$, after solving system (4.96) at the substitution of the obtained expressions for $w_i(x, t; x_j)$ and $\Pi_i(x, t; x_j)$. The integration of solution (4.93) by formula (4.75) difficulties does not cause and leads as a result to the sum of the tabulated probability integrals.

Example. Employing the given procedure can be designed the probability of disruption/separation in the various forms of dynamic effect $\lambda(t)$.

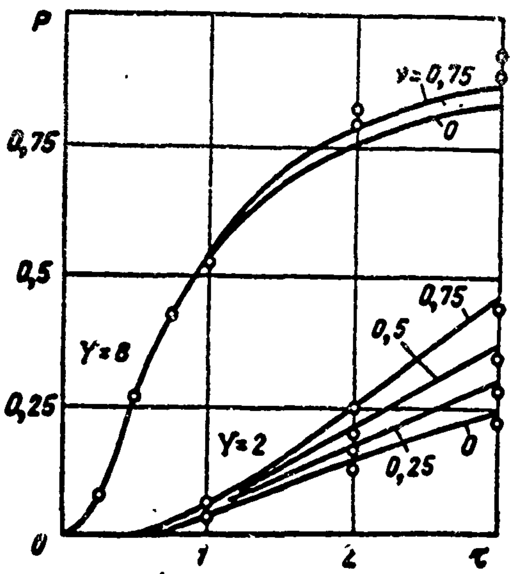

Fig. 4.5. Disruption/separation of tracking taking into account the dynamics of disturbance/perturbation.

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In particular, in Fig. 4.5 are constructed the graph/diagrams of the dependence of the probability of disruption/separation $P(\tau)$ on the generalized time of observation $\tau = KS_0t_n$ under the influence $\lambda(t)$, determined by relationships/ratios (Fig. 4.6):

$$\frac{d\lambda}{dt} = \begin{cases} V_0 & \text{при}^{(1)}\ t \leq 0, \\ V_0 + at & \text{при}^{(1)}\ 0 < t \leq t_n, \\ V_n = V_0 + at_n & \text{при}^{(1)}\ t > t_n. \end{cases}$$

Key: (1). with.

This disturbance/perturbation occurs, for example, in the automatic

range finder of radar with the target tracking which in the interval of time $0<t<t_*$ moves with the constant longitudinal acceleration.

During the calculation it was assumed that the characteristic of discriminator was symmetrical $\gamma_2=-\gamma_1=2\varphi_2=-2\varphi_1=2\varphi$ (see Fig. 4.4) and up to moment/torque t=0 of the inclusion of noise in the system is established/installed initial following error $x_0=V_0/K=0.1\varphi$. Graphs in Fig. 4.5 are constructed at the different values of parameters $Y=N_0K/S_0\varphi^2$ and $\nu=V_*/KS_0$, the first of which characterizes the relation of the power of noise and signal at the output of discriminator, the second - conservative value of dynamic following error with $t\to\infty$. The duration of the action of acceleration in the input disturbance/perturbation was received by such that $KS_0t_*=2$.

From Fig. 4.5 it follows that most strongly the dynamics of input disturbance/perturbation affects the probability of disrupting/separating the tracking with a small noise level ($Y\leq2$) and the long time of observation $\tau>2-3$. In large noise (Y=8) even essential dynamic disturbance/perturbation ($\nu=0.75$) virtually does not change the form of dependence $P(\tau)$.

Error in the method. The accuracy of the method of the compensating sources can be evaluated via the comparison of the obtained results with the results of other, more precise methods. For

example, in Fig. 4.5 points noted the values of the probability of disruption/separation, found with the method of solution of boundary-value problem (4.99) on the analog computer (see S 6.3). The comparison of results confirms the possibility of solving the boundary-value problems (at least one-dimensional) for the equation of Fokker-Planck with the fair for the practice accuracy.

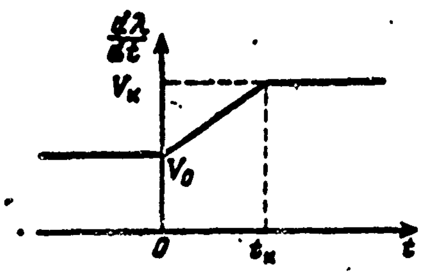

Fig. 4.6. Input dynamic effect.

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In this case can be taken into consideration the effects, connected with the complicated dynamics of input disturbance/perturbation λ(t).

Checking the accuracy of the method of the compensating sources was carried out via comparison with a series/row of other methods, for example with the method of the simulation stochastic equation in the analog and digital computers, with the method of Bubnov-Galerkin, etc. Table 4.3, in particular, gives the values of the probability of disruption/separation in first-order the system examined with $d\lambda/dt=0$, the found with the method compensations and Bubnov-Galerkin method. During the calculation it was assumed that $Y=8$, $x_*=0$.

All comparisons conducted confirm the completely satisfactory accuracy of method for the analysis of nonlinear first-order systems

up to the values of probability $P<0.6-0.8$ and time of observation $\tau\lesssim 10$ (order 10-30% on the probability of disruption/separation).

Multidimensional systems. The method of determining the probability of disruption/separation presented in the piecewise-linear systems can be spread also to the cases, when following error is the component of multidimensional Markov process [31, 84]. For some tasks which can be described by the differential second order equations, it succeeds by the method of compensation sufficient to accurately determine the probability of disrupting/separating the tracking taking into account the transiency of the conditions for the work of device/equipment. However, at the same time there are situations, when the analysis of disruption/separation in the systems of the order higher than first leads to appreciable errors in the determination of probability. Furthermore, even during the analysis of nonlinear systems with the filters of the second order the determination of the probability of disruption/separation by compensation requires conducting the great computational work, connected with the determination of the coefficients of compensation.

Table 4.3.

(1) Метод решения	(2) Вероятность срыва при τ			
	0,5	1,0	2,0	3,0
(3) Метод компенсации	0,26	0,527	0,746	0,825
(4) Метод Бубнова—Галеркина	0,26	0,505	0,808	0,925

Key: (1). Method of solution. (2). Probability of disruption/separation when τ. (3). Method of compensation. (4). Bubnov-Galerkin method.

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In such situations becomes appropriate the transition from the manual calculation to the machine. However, programming the task of calculating the coefficients of compensation is connected with the precomputation of dispersions and mathematical expectations, which are determining the fundamental solutions of the problem of Cauchy for each linear section of characteristic F(x). This task, although it does not represent fundamental difficulties, is sometimes very bulky. Therefore during the analysis of the systems of the second and higher of orders taking into account dynamics it is expedient to use TsVM for determining the probability of disruption/separation not by compensation, but one of the numerical methods, examined/considered in Chapter 6, for example, by the Monte Carlo method.

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Chapter 5.

PARTICULAR CHARACTERISTICS OF THE DISRUPTION OF TRACKING.

The probability of disrupting/separating the tracking $P(t_{н})$ for the preset time of observation $t_{н}$ is most comp[illegible]e characteristic of the phenomenon of disruption/separation. The calculation of this characteristic in many instances is hindered/hampered or requires conducting large number of computational works. Frequently, especially at the preliminary stages of the design of systems, it proves to be appropriate to prove to be from the calculation of the probability of disruption/separation, after switching over to the analysis of more particular characteristics. By such characteristics they can become, for example, mean time to the disruption/separation of tracking or critical noise level, with which the disruption/separation even on begins. The material of data they are main and dedicated to the calculation of similar characteristics.

5.1. Determination of the critical power of noise with the help of

the method of statistical linearization.

Strictly speaking, the method of statistical linearization [7], as any other method of the linearization of system, is not applied for the analysis of the disruption/separation of tracking, since for the linear system vanishes the sense of the concept itself about the disruption/separation. However, with the known stipulations and with the series/row of further limitations the method of statistical linearization can be used for the proximate analysis of the disruption/separation of tracking. This is admissible, for example, if the linearization of system is produced only in the limits of the aperture of discriminatory characteristic. In this paragraph the linearization of system will foresee itself for the determination of the series/row of the stopper facts, which associate the method of statistical linearization and approximately characterizing the stability of system. Therefore it is possible to determine the critical level of spectral density $N_{кр}$ of noise at the output of the discriminator, with which the danger of disruption/separation is still small.

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The for the first time similar method of the analysis of disruption/separation in the regulating circuits was proposed by G.

G. Sigalov and Ye. A. Yashugin [68] and B. I. Shakhtarin [69].

Theoretical analysis. Let us consider the servo system Fig. 1.2, which is located under the effect of regular $\lambda(t)$ and random $\xi(t)$ of disturbances/perturbations, the spectral density of the latter not depending on disagreement/mismatch x and constant in the passband of the ring of the automatic control: $N_{\bullet}(x) \simeq N_0$. We will be bounded also to the analysis of the systems in which dynamic disturbance/perturbation $\lambda(\tau)$ leads to constant error $\overline{x(t)} = m_x = \text{const}$.

Let us designate output potential of discriminator, caused by dynamic error, through $m_F(m_x, \sigma^2_x)$, by stressing thereby the dependence of value m_F from the mathematical expectation and the dispersion of disagreement/mismatch x(t). On the basis of the block diagram of the servo system (see Fig. 1.2) let us register the relationship/ratio, which connects mathematical expectations m_x, m_F and input dynamic disturbance/perturbation $\lambda(t)$

$$m_x = \lambda(t) - K(p)\, m_F(m_x, \sigma_x^2).$$

In steady state regular component of process at the output of discriminator is equal to

$$m_F(m_x, \sigma_x^2) = \lim_{s \to 0} \frac{s\lambda(s)}{K(s)} - \frac{m_x}{K(0)}, \qquad (5.1)$$

where $\lambda(\sigma)$, K(s) - converted according to Laplace input disturbance/perturbation $\lambda(t)$ and operator K(p).

When system possesses astaticism of the n order, i.e.

$$K(p)=\frac{K}{p^{n}}\frac{\mathcal{B}_1(p)}{\mathcal{C}_2(p)},\ \mathcal{B}_1(0)=1,$$

dynamic error m_x is constant, if disturbance/perturbation takes the form of polynomial not older than the n degree

$$\lambda(t)=\lambda_0+\lambda_1 t+\ldots+\lambda_n t^n.$$

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In this case to the linearized system

$$m_x=\frac{\lambda_n}{K\cdot K_0(m_x,\ \sigma_x^2)},\qquad (5.2)$$

where $K_0(m_x,\ \sigma_x^2)$ — equivalent mutual conductance of discriminator, introduced according to the method of statistical linearization and which considers the passage only of regular component. Introducing the equivalent slope/transconductance of discriminator $K_1(m_x,\ \sigma_x^2)$ for the central random component, let us register expression for the dispersion in the linearized system

$$\sigma_x^2=\frac{N_0}{f\,[K_1(m_x,\ \sigma_x^2),\ K(p)]},\qquad (5.3)$$

where

$$f\,[K_1,\ K(p)]=\left[\frac{1}{2\pi}\int_0^\infty\left|\frac{K(j\omega)}{1+K_1(m_x,\ \sigma_x^2)\,K(j\omega)}\right|^2 d\omega\right]^{-1};$$

K(jω) - the complex gear ratio/transmission factor of feedback loop.

Let us represent (5.2) and (5.3) in the form of system of

equations

$$\left.\begin{array}{l}\lambda_{n}=\varphi\cdot m_{x},\\ N_{0}=f\cdot\sigma_{x}^{2},\end{array}\right\}\qquad(5.4)$$

where $\varphi=K\cdot K_{0}(m_{x},\sigma_{x}^{2}),\ f=f[K_{1}(m_{x},\sigma_{x}^{2}),K(p)]$.

If parameters λ_n and N_0 of input disturbances/perturbations are such, that the system of tracking is located on the face of disruption/separation, then their small variations lead to large changes in conservative values m_x and σ_x^2. Let us give increase $\Delta\lambda_n$ to parameter λ_n. In accordance with (5.2) and (5.3) this will produce increases in mathematical expectation m_x and dispersion σ_x^2. System (5.4) of signs the form

$$\left.\begin{array}{c}\lambda_{n}+\Delta\lambda_{n}=(m_{x}+\Delta m_{x})\times\\ \times\varphi[K_{0}(m_{x}+\Delta m_{x},\ \sigma_{x}^{2}+\Delta\sigma_{x}^{2}),\ K(p)],\\ N_{0}=(\sigma_{x}^{2}+\Delta\sigma_{x}^{2})\times\\ \times f[K_{1}(m_{x}+\Delta m_{x},\sigma_{x}^{2}+\Delta\sigma_{x}^{2}),\ K(p)].\end{array}\right\}\qquad(5.5)$$

Is decomposed nonlinear functions φ and f in the Taylor series according to degrees Δm_x and $\Delta\sigma_x^2$ and we will be bounded to the linear terms of expansion.

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After the subtraction of steady-state values (5.4) we will obtain for

the increases system of equations:

$$\left.\begin{aligned}\left(\varphi+m_x\frac{\partial\varphi}{\partial m_x}\right)\Delta m_x+m_x\frac{\partial\varphi}{\partial\sigma_x^2}\Delta\sigma_x^2=\Delta\lambda_x,\\ \sigma_x^2\frac{\partial f}{\partial m_x}\Delta m_x+\left(f+\sigma_x^2\frac{\partial f}{\partial\sigma_x^2}\right)\Delta\sigma_x^2=0.\end{aligned}\right\}\quad(5.6)$$

Loss of stability occurs in such a case, when the determinant of system of equations (5.6) becomes zero. Hence, taking into account the implicit dependence of functions φ and f on m_x and σ_x^2, it is possible to register the stall conditions of tracking [68]

$$\left(\varphi+m_x\frac{\partial\varphi}{\partial K_0}\frac{\partial K_0}{\partial m_x}\right)\left(f+\sigma_x^2\frac{\partial f}{\partial K_1}\frac{\partial K_1}{\partial\sigma_x^2}\right)-$$
$$-m_x\sigma_x^2\frac{\partial\varphi}{\partial K_0}\frac{\partial K_0}{\partial\sigma_x^2}\frac{\partial f}{\partial K_1}\frac{\partial K_1}{\partial m_x}=0.\quad(5.7)$$

In the particular case of the absence of dynamic error $(m_x=0)$ relationship/ratio (5.7) is simplified and takes the form

$$\frac{\partial\sigma_x^2}{\partial N_0}=0.\quad(5.8)$$

The direct use of relationship/ratio (5.7) for the practical calculations is frequently connected with the bulky transformations. Therefore to more conveniently use the graphic method of determining the critical noise level.

Graphic method. Let us clarify the graphic method of determining the critical noise level based on specific example.

Let there be the servo system of phase automatic frequency control with one integrator in the feedback loop K(p)=K/p (see Fig. 3.5) and characteristic of the discriminator

$$F(x)=A\sin ax. \qquad (5.9)$$

On the system functions dynamic disturbance/perturbation $\lambda(t)=\lambda_2 t$ and white noise $\xi(t)$ with a spectral density of N_0. It is necessary to determine the critical value of spectral density N_0. It is necessary to determine the critical value of spectral density $N_{кр}$.

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The solution of problem let us begin from the calculation of coefficients $K_0(m_x, \sigma_x^2)$ and $K_1(m_x, \sigma_x^2)$ the transmissions of the linearized system for constant and random components. Following the method of statistical linearization assuming that the following error x(t) is distributed according to the law, close to the normal, we have

$$K_0(m_x, \sigma_x^2)=\frac{1}{m_x}\int_{-\infty}^{\infty}F(x)\,w(x)\,dx=\frac{A}{\sqrt{2\pi}\sigma_x m_x}\times$$

$$\times\int_{-\infty}^{\infty}\sin(ax)\exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2}\right]dx=$$

$$=\frac{A}{m_x}\sin(am_x)\exp\left(-\frac{a^2\sigma_x^2}{2}\right), \qquad (5.10)$$

$$K_1(m_x, \sigma_x^2)=\frac{\partial(m_x K_0)}{\partial m_x}=Aa\cos(am_x)\exp\left(-\frac{a^2\sigma_x^2}{2}\right). \qquad (5.11)$$

Further calculation requires the concrete definition of the parameters of system. Let K=1 rad/(V·s), A=1 V, a=1 rad^{-1}, λ_1=0.3 rad/s.

Taking into account that $m_y = K_0 m_x$ and by using relationship/ratio (5.10), let us construct auxiliary family of curves $m_y = m_y(m_x)$ (Fig. 5.1) at the different values of dispersion σ_x^2.

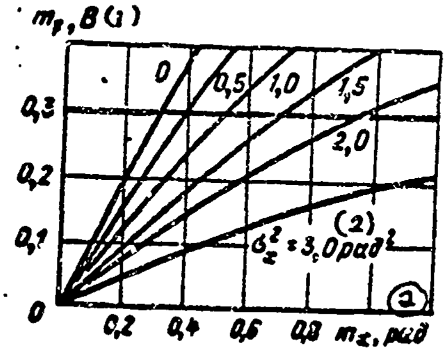

Fig. 5.1.

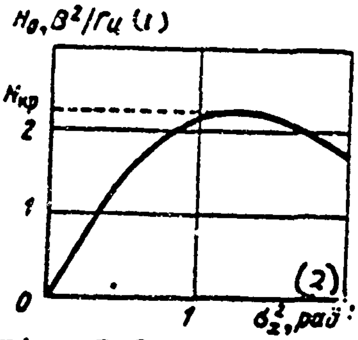

Fig. 5.2.

Fig. 5.1. Auxiliary graphs for calculating critical power of noise.

Key: (1). V. (2). rad.

Fig. 5.2. Determination of critical power of noise.

Key: (1). V²/Hz. (2). rad.

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Conservative value of dynamic error, led to the output of discriminator, on the basis (5.1) is equal

$$m_p(m_x, \sigma_\omega^2) = \frac{\lambda_1}{K}. \qquad (5.12)$$

Equation (5.12) is graphically the horizontal line (Fig. 5.1)

whose ordinat· in this case is equal to $m_F = 0{,}3$ V. For the points of interse·tion of the constructed dependences with the straight line let us find the appropriate values of coefficient $K_1(m_x, \sigma_x^2)$ in formula (5.11). Taking into account that the second equation of system (5.4) in the case in question takes the form

$$N_0 = \frac{4\sigma_x^2 K_1(m_x, \sigma_x^2)}{K},$$

le· us d· mine spectral density for each obtained value $K_1(m_x, \sigma_x^2)$ and let us construct the graph/diagram of dependence $N_0 = f(\sigma_x^2)$ (Fig. 5.2).

As can be seen from Fig. 5.2, with smalls of the level of input disturbance/perturbation the variance of error of tracking is in effect proportional to spectral density. In this case process x(t) is developed in essence in the linear section of the characteristic of discriminator. With an increase in the spectral density the proportionality is broken and near $N_0 = N_{кр}$ the rate of the build-up/growth of dispersion becomes infinite, i.e., system loses stability, is observed the disruption/separation of tracking. Thus, the critical value of spectral density in the example in question comprises $N_{кр} \approx 2{,}24$ V²/Hz.

It is interesting to consider, with what probability occurs the disruption/separation of tracking in the system in question under the

effect on it of the noise whose spectral density is equal to critical. For this servo system it is possible to simulate/model in the digital computer and by the Monte Carlo method to determine the probability of disrupting/separating the tracking for the preset time of observation. The graph/diagram of the obtained dependence of the probability of disruption/separation on the spectral noise density with the time of observation $t_н = 5\frac{\Delta\omega}{K}$ is depicted in Fig. 5.3.

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As can be seen from graph, the obtained estimation of the critical power of noise determines sufficiently well conditions, with which the disruption/separation of tracking becomes dangerous.

Fig. 5.4 depicts the dependence of critical spectral density $N_{кр}$ in the system in question on the value of dynamic disturbance/perturbation λ_1. Curve is constructed with the help of the graphic method of analysis.

Analysis of the system of the second order. Let us consider one additional example, which has larger practical value. Let the feedback loop of regulating circuit have a gear ratio/transmission factor

$$K(p) = \frac{K(1 + pT_1)}{p^2}, \qquad (5.13)$$

and the characteristic of discriminator is approximated by the dependence

$$F(x)=Sx\exp\left(-\frac{a^2x^2}{2}\right). \qquad (5.14)$$

Input dynamic disturbance/perturbation $\lambda(t)$ let us place equal to $\lambda(t)=\lambda_0+\lambda_1 t$.

Since the system possesses astaticism of the second order, conservative value of following error is equal to zero.

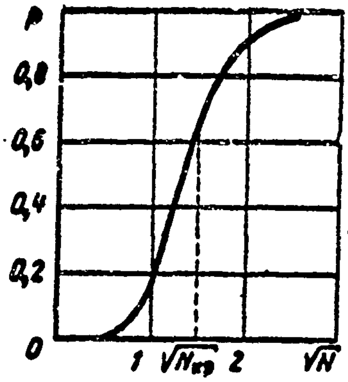

Fig. 5.3.

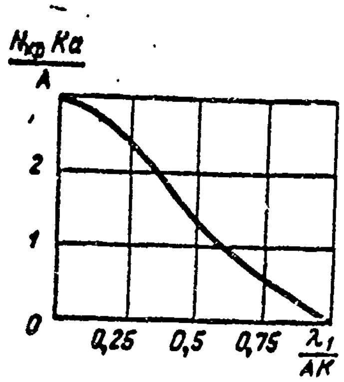

Fig. 5.4.

Fig. 5.3. Dependence of probability of disruption/separation on noise level at power of close one to critical.

Fig. 5.4. Dependence of critical noise level on dynamic disturbance/perturbation and to first-order system.

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For the gear ratio/transmission factor of discriminator on random component we have

$$K_1(m_x, \sigma_x^2) = \frac{1}{\sqrt{2\pi}\sigma_x^3} \int_{-\infty}^{\infty} (x - m_x) F(x) \times$$

$$\times \exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2}\right] dx = S(1+a^2\sigma_x^2)^{-3/2}. \qquad (5.15)$$

The variance of error of tracking in the linearized system on

the basis (5.3) is equal to

$$\sigma_{\bullet}^{2}=N_{\bullet}\frac{1+K\cdot K_{1}(m_{\bullet},\sigma_{x}^{2})T_{1}^{2}}{4T_{1}K_{1}^{2}(m_{\bullet},\sigma_{x}^{2})}, \tag{5.16}$$

whence taking into account (5.15) we obtain

$$N_{\bullet}=\frac{4T_{1}\sigma_{x}^{2}S^{2}}{(1+a^{2}\sigma_{x}^{2})^{3/2}\,[(1+a^{2}\sigma_{x}^{2})^{3/2}+KST_{1}^{2}]}. \tag{5.17}$$

For determining the critical spectral density $N_{кр}$ it is necessary to find such values of $N_{\bullet}$, with which the derivative of expression (5.17) on dispersion σ_{x}^{2} would become zero. Calculations in this case to more conveniently produce graphically. The results of calculations are shown in Fig. 5.5, where are constructed the dependences of dimensionless spectral density $N_{\bullet}=N_{кр}a^{2}\sqrt{KS}/4S^{2}$ on the generalized parameter of system KST^{2}_{1}. In Fig. 5.5 is noticeable not pronounced optimum, which is observed with $KST^{2}_{1}\approx 2.5$. The experimental check confirms the presence of optimum in the system with astaticism of the second order, although at smaller value of KST^{2}_{1}.

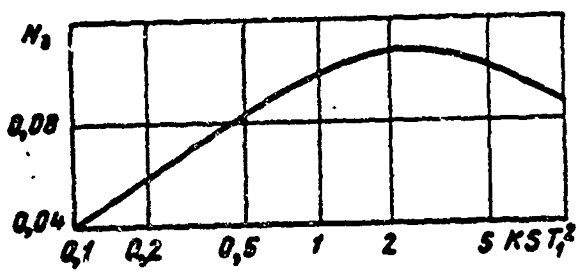

Fig. 5.5. Critical spectral density in the system with astaticism of the second order.

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Conclusions/outputs. The method of the analysis of nonlinear regulating circuits examined makes it possible to approximately consider noise level with which the mode/conditions of tracking becomes unreliable. As showed the experimental checks, method gives the correct estimation of order of magnitude $N_{кр}$ which has the vital importance with they are approximate the calculations of regulating circuits.

The advantage of method is its comparative simplicity and the possibility of method is its comparative simplicity and possibility of the analysis of systems virtually with any filters and discriminatory characteristics. When the fluctuating characteristic of discriminator depends on disagreement/mismatch x, the method

presented can be used after the statistical averaging of characteristic $N_{e}(x)$ according to formula (3.20).

Deficiencies/lacks in the method include its comparatively low accuracy and impossibility to obtain the temporary/time and statistical characteristics of the phenomenon of disruption/separation. Therefore one ought not to use this method for the determination of the thin effects, connected with the work of the follower (for example, for determining the optimum parameters of discriminator and filter in cases when optimum it is expressed weakly). Due to the errors, inherent in the method of statistical linearization, in these cases can be allowed noticeable errors.

5.2. Determination of critical stall conditions on the basis of the equation of Pontriagin.

The determination of critical spectral density N_{*}, with the help of the method of statistical linearization is frequently connected with the cumbersome calculations. Therefore let us consider one additional method [88] of determining the critical power of noise, valid for the nonlinear systems of first order. At the basis of method lies/rests the fact that the first approximation for the probability of disruption/separation, found with Bubnov-Galerkin method, under some conditions leads to the results, which contradict

the physical basis of phenomenon. For the first time to this is converted attention in the work of I. A. Bol'shakov [46].

The probability of the absence of disruption/separation in first-order fixed system satisfies the equation of Pontriagin

$$\frac{\partial U(x_0, t)}{\partial t} = -f(x_0)\frac{\partial U}{\partial x_0} + \frac{B(x_0)}{2}\frac{\partial^2 U}{\partial x_0^2}, \qquad (5.18)$$

moreover

$$U(\pm\gamma, t) = 0.$$

In accordance with Bubnov-Galerkin method (see § 4.2) let us find first approximation for functions $U(x_0, t)$ in the form

$$U_1(x_0, t) = C\psi(t)\varphi(x_0).$$

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Assuming/setting $\varphi(x_0) = \cos\frac{\pi x_0}{2\gamma}$, we will obtain

$$\psi(t) = \exp\left\{-\frac{\pi^2 t}{8\gamma^3}\int_{-\gamma}^{\gamma} B(x_0)\cos^2\frac{\pi x_0}{2\gamma}dx_0 + \right.$$
$$\left. + \frac{\pi t}{4\gamma^2}\int_{-\gamma}^{\gamma} f(x_0)\sin\frac{\pi x_0}{\gamma}dx_0\right\}.$$

From the physical considerations it is clear that the function $\psi(t)$ must decrease in the time. This is possible only in such a case, when is satisfied the condition

$$\frac{\pi}{2\gamma}\int_{-\gamma}^{\gamma} B(x_0)\cos^2\frac{\pi x_0}{2\gamma}dx_0 > \int_{-\gamma}^{\gamma} f(x_0)\sin\frac{\pi x_0}{\gamma}dx_0. \qquad (5.19)$$

In this case it proves to be that the amount of the minimum power of noise, at which inequality (5.19) still is fulfilled, is very close to the critical, computed from the method statistical linearization. Thus, the determination of the critical stall conditions of tracking in the nonlinear first-order systems can be reduced to the solution of the algebraic equation

$$\int_{-\gamma}^{\gamma} \beta(x_0) \cos^2 \frac{\pi x_0}{2\gamma} dx_0 = \frac{2\gamma}{\pi} \int_{-\gamma}^{\gamma} f(x_0) \sin \frac{\pi x_0}{\gamma} dx_0. \qquad (5.20)$$

To compute definite integrals in (5.20), as a rule, is not difficult; therefore the calculation of the critical stall conditions of tracking on the basis of equation (5.20) can prove to be simpler than the calculation according to the method of statistical linearization.

Example. Let us consider servo system with the integrator in the feedback loop (see Fig. 3.5). Stochastic differential equation of this system takes form (3.25). The input dynamic disturbance/perturbation $\lambda(t)$= const, and the characteristic of the discriminator

$$F(x) = \begin{cases} Sx \ \overset{(1)}{\text{при}} \ |x| \leqslant 1, \\ 0 \ \text{при} \ |x| > 1. \end{cases}$$

Key: (1). with.

Let the spectral density of the white noise, led to the output

of discriminator, not depend on following error, i.e., $N_0(x) = N_0$. Then under the done assumptions

$$f(x_0) = KSx_0, \ B(x_0) = \frac{K^2N_0}{2}$$

and equation (5.20) takes the form

$$\frac{\pi}{4}K^2N_c = \frac{2}{\pi}SK. \qquad (5.21)$$

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Hence critical spectral noise density

$$N_{кр} = \frac{8S}{\pi^3 K} \approx 0.81\frac{S}{K}. \qquad (5.22)$$

Calculation according to the method of statistical linearization in this case gives $N_{кр} \approx 0.86S/K$, which is close to (5.22).

Let us note that the method of determining the critical spectral density examined is valid in cases when the fluctuating characteristic of discriminator is not the constant ($N_0(x) \neq$ const). Some examples of the determination of critical stall conditions by the method presented are also in work [88].

5.3. The time characteristics of the disruption/separation of tracking.

In many practical cases it is necessary to know the probabilistic characteristics of time interval, which passed from the beginning of observation to the disruption/separation. Total characteristic of time of disruption/separation is its density of distribution $W_{x_0}(T)$. Regarding $W_{x_0}(T)\Delta t$ indicates probability that in the system with initial conditions x_0, which occurred at moment/torque t=0, the disruption/separation of tracking will occur in the interval of time $T-\frac{\Delta t}{2}<t<T+\Delta t/2$.

Examining the integral law of time allocation to the disruption/separation

$$P(x_0, T)=\int_0^T W_{x_0}(\tau)\,d\tau, \quad (5.23)$$

let us note that

$$P(x_0, T)=1-\int_\Omega w(x, t;\ x_0)\,dx, \quad (5.24)$$

where $w(x, t; x_0)$ - the probability density of the transition of random process of x(t) for time T from the phase state x_0 into the state χ. Function $w(x, t; x_0)$ can be determined by the method of solution of the corresponding boundary-value problem for the equation of Fokker-Planck in the n-dimensional phase space Ω. From (5.24) it follows that the integral density of distribution of time to the disruption/separation coincides with the solution of boundary-value problem for the equation of Pontriagin (2.78).

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The determination of the density of distribution of time to the disruption/separation by the method of solution of the equation of Fokker-Planck or Pontriagin can it is carried out by the same methods which were examined during the analysis of the probability of disrupting/separating the tracking. As a rule, these methods are very labor-consuming. However, in many instances it suffices to know less total characteristics. Frequently, for example, it is possible to be bounded only to the determination of several first moments of time to the disruption/separation.

Mean time to the disruption/separation. For the first moment of time to the disruption/separation it is possible to register

$$m_1(x_0)=\int_0^\infty T W_{x_0}(T)\,dT=\int_0^\infty T\frac{\partial P(x_0, T)}{\partial T}\,dT. \qquad (5.25)$$

Introducing the probability of tracking $U(x_0, T)=1-P(x_0, T)$ and computing integral in (5.25) in parts taking into account the fact

that $\lim_{T\to\infty} P(x_0, T)=1$ we will obtain

$$m_1(x_0)=\int_0^\infty U(x_0, T)\,dT. \qquad (5.26)$$

Using the obtained relationship/ratio, let us form an equation for the mean time to the disruption/separation, understanding under the disruption/separation the first output of following error beyond the limits of the aperture of discriminatory characteristic. Let the equation of Pontriagin for probability $U(x_0, T)$ of the absence of disruption/separation for time T take the form

$$\frac{\partial U}{\partial T}=\sum_{i=1}^{n} A_i(x_0)\frac{\partial U}{\partial x_{0i}}+\frac{1}{2}\sum_{i,j=1}^{n} B_{ij}(x_0)\frac{\partial^2 U}{\partial x_{0i}\partial x_{0j}}, \quad (5.27)$$

where x_0 - n-dimensional vector, which characterizes the initial state of system $x_0=(x_{01}, x_{02}, \ldots, x_{0n})$. The coefficients of removal/drift $A_i(x_0)$ and diffusion $B_{ij}(x_0)$ do not depend on time, which is correct for the systems, which have the constant parameters and which are located under the stationary effects.

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After integrating each term of equation (5.27) for by the variable/alternating T from zero to and taking into account that

$$\int_0^\infty \frac{\partial U(x_0, T)}{\partial T}\,dT=U(x_0, T)\Big|_0^\infty=-1,$$

on the basis (5.26) we will obtain differential equation for the mean

time to the disruption/separation

$$\frac{1}{2}\sum_{i,j=1}^{n} B_{ij}(x_0)\frac{\partial^2 m_1(x_0)}{\partial x_{0i}\partial x_{0j}}+\sum_{i=1}^{n} A_i(x_0)\frac{\partial m_1(x_0)}{\partial x_{0i}}+1=0. \quad (5.28)$$

Equation (5.28) is for the first time obtained by L. S. Pontriagin in work [35], in connection with which it they occasionally refer to as second equation of Pontriagin.

During the determination of mean time to the disruption/separation equation (5.28) is supplemented by the boundary conditions

$$m_1(x_0)|_{x_0\in G^*}=0, \quad (5.29)$$

where G^* is regular part of the boundary of the region of tracking Ω in the phase space for the equations of Pontriagin (see § 2.6).

If following error $x(t)$ is one-dimensional Markov process, then (5.28) it is converted into the ordinary differential equation

$$\frac{1}{2}B(x_0)\frac{d^2m_1}{dx_0^2}+A(x_0)\frac{dm_1}{dx_0}+1=0 \quad (5.30)$$

with the uniform boundary conditions

$$m_1(\gamma_1)=m_1(\gamma_2)=0, \quad (5.31)$$

where γ_1, γ_2 - boundaries of the aperture of discriminatory characteristic.

Equation (5.30) belongs to the class of linear differential equations with the variable coefficients. Its solution is written out in general form.

Actually/really, after designating $y=dm_1/dx_0$, we will obtain

$$\frac{dy}{dx_0}+2\frac{A(x_0)}{B(x_0)}y+\frac{2}{B(x_0)}=0. \tag{5.32}$$

Let us introduce the new functions u and V so, in order to y=uv. In this case (5.32) take the form

$$v\frac{du}{dx_0}+u\frac{dv}{dx_0}+2\frac{A(x_0)}{B(x_0)}uv+\frac{2}{B(x_0)}=0. \tag{5.33}$$

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Let us select v in such a way that it would be performed the equality

$$u\frac{dv}{dx_0}+2\frac{A(x_0)}{B(x_0)}uv=0, \tag{5.34}$$

hence we obtain

$$v=Ce^{-\varphi(x_0)},\ \varphi(x_0)=2\int\frac{A(x_0)}{B(x_0)}dx_0. \tag{5.35}$$

From (5.33) taking into account (5.34) we obtain equation for determining the function u:

$$v\frac{du}{dx_0}+\frac{2}{B(x_0)}=0.$$

Taking into account of expression (5.35), we find the solution of the latter/last equation

$$u = -\frac{2}{C}\int_{C_1}^{x_0}\frac{1}{B(z)}e^{\varphi(z)}dz.$$

Taking into account that $y = dm_1/dx_0 = uv$, finally we will obtain

$$m_1(x_0) = -2\int_{C_2}^{x_0} e^{-\varphi(x)}\int_{C_1}^{x}\frac{1}{B(z)}e^{\varphi(z)}dz dx.$$

Determining integration constant from boundary conditions (5.31), let us register the resultant expression for the mean time of reaching/achievement of boundaries γ_1, γ_2 by the one-dimensional Markov process

$$m_1(x_0) = \left\{\left[\int_{x_0}^{\gamma_2} e^{-\varphi(x)}\int_{\gamma_1}^{x}\frac{2}{B(z)}e^{\varphi(z)}dz dx\right]\int_{\gamma_1}^{x_0} e^{-\varphi(x)}dx - \right.$$
$$\left.-\left[\int_{\gamma_1}^{x_0} e^{-\varphi(x)}\int_{\gamma_1}^{x}\frac{2}{B(z)}e^{\varphi(z)}dz dx\right]\int_{x_0}^{\gamma_2} e^{-\varphi(x)}dx\right\}\times$$
$$\times\left[\int_{\gamma_1}^{\gamma_2} e^{-\varphi(x)}dx\right]^{-1}. \quad (5.36)$$

Moments/torques of higher orders. In the case of one-dimensional Markov process comparatively easily are written/recorded the equations for the moments of time to the disruption/separation of the order higher than first. Let the equation of Pontriagin for probability $P(x_0, T)$ of achieving the boundaries γ_1, γ_2 for time T take the form

$$\frac{\partial P(x_0, T)}{\partial T} = A(x_0)\frac{\partial P}{\partial x_0} + \frac{1}{2}B(x_0)\frac{\partial^2 P}{\partial x_0^2}.$$

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Differentiating it on time T and taking into account that the density of distribution of time to the disruption/separation on the basis (5.23) is equal to

$$W_{x_0}(T) = -\frac{\partial P(x_0, T)}{\partial T}, \qquad (5.37)$$

we will obtain

$$\frac{\partial W_{x_0}(T)}{\partial T} = A(x_0)\frac{\partial W_{x_0}(T)}{\partial x_0} + \frac{1}{2}B(x_0)\frac{\partial^2 W_{x_0}(T)}{\partial x_0^2}. \qquad (5.38)$$

Let us introduce the characteristic function of time to the disruption/separation of the tracking

$$\theta(\nu, x_0) = \int_0^\infty W_{x_0}(T)\, e^{j\nu T}\, dT. \qquad (5.39)$$

Let us multiply each term of equation (5.38) on $e^{j\nu T}$ and let us produce integration in accordance with (5.39). As a result we will obtain

$$\frac{1}{2}B(x_0)\frac{d^2\theta}{dx_0^2} + A(x_0)\frac{d\theta}{dx_0} + j\nu\theta = 0. \qquad (5.40)$$

It is known that the characteristic function $\theta(\nu, x_0)$ can be represented by Maclaurin series:

$$\theta(\nu, x_0) = 1 + \sum_{k=1}^{\infty}\frac{m_k(x_0)}{k!}(j\nu)^k. \qquad (5.41)$$

In this case $m_k(x_0)$ — the moment/torque of the k order of time to the disruption/separation of tracking. Substituting expansion (5.41) in equation (5.40) and equalizing coefficients with identical degrees ($j\nu$), we will obtain the following recurrent equation for the moment/torque of the k order

$$\frac{1}{2}B(x_0)\frac{d^2m_k(x_0)}{dx_0^2}+A(x_0)\frac{dm_k(x_0)}{dx_0}+km_{k-1}(x_0)=0,$$
$$k=1,\ 2,\ldots \tag{5.42}$$

moreover $m_0(x_0)=1$. Equations (5.42) must have solutions under the boundary conditions

$$m_k(\gamma_1)=m_k(\gamma_2)=0. \tag{5.43}$$

From (5.42) follows, in particular, equation (5.30) for the mean time to the disruption/separation.

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Dispersion of time to the disruption/separation. Equation for the dispersion of time to the disruption/separation on the basis (5.42) takes the form

$$\frac{1}{2}B(x_0)\frac{d^2\mathcal{D}}{dx_0^2}+A(x_0)\frac{d\mathcal{D}}{dx_0}+B(x_0)\left(\frac{dm_1}{dx_0}\right)^2=0, \tag{5.44}$$

where

$$\mathcal{D}(x_0)=m_2(x_0)-m_1^2(x_0).$$

moreover

$$\Phi(\gamma_1) = \Phi(\gamma_2) = 0. \tag{5.45}$$

Let us consider some examples of the calculation of the time characteristics of the disruption/separation of tracking.

1. First-order system with "linear" discriminator. Let the follower have linear with slope/transconductance S the characteristic of discriminator in the limits of aperture $-\gamma<x<\gamma$ (see Fig. 3.1) and filter in the feedback loop

$$K(p) = \frac{K}{1+pT}. \tag{5.46}$$

Stochastic equation, which describes the behavior of the analyzed system, takes the form

$$T\frac{dx}{dt} + x(KS+1) = \lambda(t) + T\frac{d\lambda}{dt} - K\sqrt{N_0}\xi(t),$$
$$-\gamma < x < \gamma, \tag{5.47}$$

where $\lambda(t)$ - input dynamic effect.

The example in question is completely realistic.

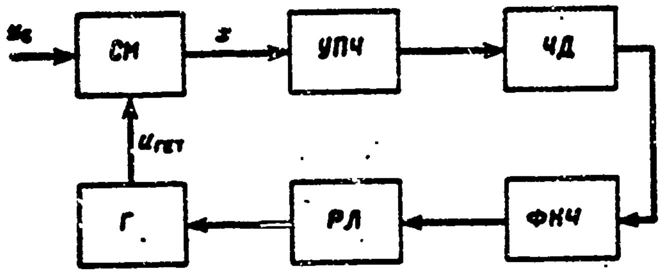

Fig. 5.6. The functional diagram of the particular self-alignment: SM - mixer; UPCh - amplifier of intermediate frequency; ChD - frequency discriminator; FNCh - low-pass filter; RL - reactance tube; Г - adjustable/tuneable heterodyne.

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By equation (5.47) is described, for example, the behavior of the system of frequency self-alignment (Fig. 5.6) with filter (5.46) when the passband of UPCh already of the staggering of frequency discriminator. In this case the characteristic of discriminator practical is linear in the limits of the passband Δf of UPCh. If detuning x between the signal frequencies and heterodyne exceeds half of band UPCh $|x|>\Delta f/2=\gamma$, then signal to the entrance of discriminator does not pass and system is broken; therefore points $\pm\gamma$ can be considered the absorbing boundaries.

Examining the case $\lambda(\tau)=\lambda=$ const, on the basis (5.47) let us

register equation for the mean time to the disruption/separation

$$\frac{K^2N_0}{4T^2}\frac{d^2m_1}{dx_0^2}-\left(ax_0-\frac{\lambda}{T}\right)\frac{dm_1}{dx_0}+1=0, \quad (5.48)$$

moreover

$$a=\frac{1+KS}{T}, \quad x_0=x(t=0), \quad m_1(\pm\gamma)=0.$$

Introducing the new variable/alternating

$$y=x_0-\frac{\lambda}{aT}, \quad (5.49)$$

let us represent (5.48) in the form

$$\frac{K^2N_0}{4T^2}\frac{d^2m_1}{dy^2}-ay\frac{dm_1}{dy}+1=0, \quad (5.50)$$

$$m_1(y=\gamma_{1(2)})=0, \quad \gamma_1=-\gamma-\frac{\lambda}{aT},$$

$$\gamma_2=\gamma-\frac{\lambda}{aT}.$$

Equation (5.50) is a special case of equation (5.30), in which one should assume

$$B(x_0)=\frac{K^2N_0}{2T^2}, \quad A(x_0)=-ax_0. \quad (5.51)$$

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On the basis of (5.36), let us find expression for the mean time to the disruption/separation in the example

$$m_1(x_0)=\frac{\sqrt{\pi}}{a}\Big\{I(\varphi_1,\varphi_0)\int_{\varphi_0}^{\varphi_2}[\Phi(x)-\Phi(\varphi_1)]\,e^{x^2}dx-$$

$$-I(\varphi_0,\varphi_2)\int_{\varphi_1}^{\varphi_0}[\Phi(x)-\Phi(\varphi_1)]\,e^{x^2}dx\Big\}\,I^{-1}(\varphi_1,\varphi_2). \quad (5.52)$$

$$I(z_1, z_2)=\int_{z_1}^{z_2} e^{x^2}dx, \quad \Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx;$$

Where

$$\varphi_0=\frac{1}{\sqrt{2}\sigma}\left(x_0-\frac{\lambda}{1+KS}\right);$$

$$\varphi_1=-\frac{1}{\sqrt{2}\sigma}\left(\gamma+\frac{\lambda}{1+KS}\right);$$

$$\varphi_2=\frac{1}{\sqrt{2}\sigma}\left(\gamma-\frac{\lambda}{1+KS}\right);$$

$$\sigma^2=\frac{K^2N_0}{4T(1+KS)}.$$

With a small dynamic error $\lambda \ll \gamma(1+KS)$ we have

$$m_1\approx\frac{\sqrt{\pi}}{\alpha}\int_{\varphi_0}^{\varphi_2}\Phi(x)\,e^{x^2}dx, \tag{5.53}$$

$$\varphi_0=\frac{x_0}{\sqrt{2}\sigma}, \quad \varphi_2=-\varphi_1=\frac{\gamma}{\sqrt{2}\sigma}.$$

According to formula (5.53) in Fig. 5.7 are constructed the graph/diagrams of the dependence of dimensionless mean time to the disruption/separation $\alpha\mu_1$ on generalized parameter $\gamma/\sqrt{2}\sigma$ at the different values of the initial error $X=x_0/\gamma$.

Let us pass to determining the dispersion of time to the disruption/separation.

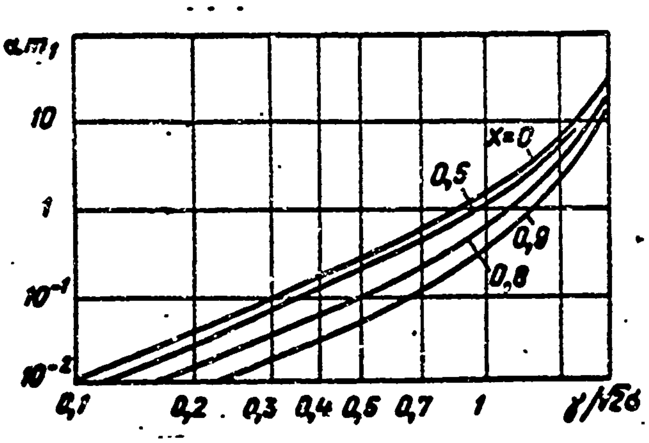

Fig. 5.7. Mean time to the disruption/separation in first-order system.

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For this it is necessary to solve boundary-value problem for ordinary differential equation (5.44) with boundary conditions (5.45). The solution of equation (5.44) can be found in general form with the method which was used during the solution of equation (5.30). the result of solution taking into account conditions (5.45) takes the form

$$\mathscr{D}(x_0) = \int_{\gamma_1}^{x_0} e^{-\varphi(z)} \int_{\gamma_1}^{z} \eta(y)\, e^{\varphi(y)}\, dy\, dz - C \int_{\gamma_1}^{x_0} e^{-\varphi(z)}\, dz, \quad (5.54)$$

where $$C = \int_{\gamma_1}^{\gamma_2} e^{-\varphi(z)} \int_{\gamma_1}^{z} \eta(y)\, e^{\varphi(y)}\, dy\, dz \left[\int_{\gamma_1}^{\gamma_2} e^{-\varphi(z)}\, dz\right]^{-1};$$

$$\varphi(x) = 2\int \frac{A(x)}{B(x)}\, dx, \quad \eta(y) = 2\left(\frac{dm_1}{dx_0}\right)^2;$$

$m_1(x_0)$ - mean time to the disruption/separation of tracking as the function of the initial error x_0. In the general case of $m_1(x_0)$ it is determined by dependence (5.36).

We will be bounded to the examination of a special case of symmetrical boundaries $\gamma_2=-\gamma_1=\gamma$ with the low value of dynamic error $\lambda\ll\gamma(1+KS)$. With satisfaction of these conditions mean time to the disruption/separation is determined by expression (5.53); therefore

$$\tau(y)=\frac{\pi}{\alpha^2\sigma^2}\Phi^2\left(\frac{y}{\sqrt{2}\sigma}\right)\exp\left(\frac{y^2}{\sigma^2}\right). \qquad (5.55)$$

Taking into account the concrete/specific/actual form of coefficients (5.51) of initial equation, on the basis (5.54) and (5.55) after some transformations we will obtain the following expression for the dispersion of time to the disruption/separation during the zero initial disagreement/mismatch:

$$D(0)=\frac{2\pi}{\alpha^2}\int_0^{\gamma/\sqrt{2}\sigma} e^{x^2}\int_t^x \Phi^2(t)\, e^{t^2}\, dt dx. \qquad (5.56)$$

The dependence of dimensionless dispersion $\alpha^2 D(0)$ on generalized parameter $\gamma/\sqrt{2}\sigma$ is constructed according to formula (5.56) in Fig.

5.8. At the low values $\gamma/\sigma \ll 1$ expression (5.56) considerably is simplified

$$\varnothing(0) \approx \frac{1}{6\alpha^2}\left(\frac{\gamma}{\sigma}\right)^4. \qquad (5.57)$$

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As showed numerical checking, latter/last formula gives the possibility to calculate the dispersion of the time to the disruption/separation with the accuracy not less than 20%, if $\gamma/\sqrt{2}\sigma \lesssim 0{,}5$.

The expressions obtained in this example for the mean time and the dispersion of time to the disruption/separation of tracking can be used also for the approximate determination of the time characteristics of disruption/separation in the nonlinear first-order systems, if we preliminarily produce their linearization.

When the feedback loop of control system has instead of (5.46) operational gear ratio/transmission factor $K(p)=K/p$, the obtained relationships/ratios and graphs will remain in the force, if we consider that $\alpha=KS$, $\sigma^2=KN_0/4S$, $\gamma_2=-\gamma_1-\gamma$.

2. First-order system with rectangular characteristic of discriminator. Let us determine mean time to the

disruption/separation of tracking in the system, which has the discriminatory characteristic:

$$F(x)=\begin{cases} A \text{ при}^{(1)}\ 0<x<\gamma, \\ -A \text{ при}^{(1)}\ -\gamma<x<0, \\ 0 \text{ при } |x|>\gamma \end{cases}$$

Key: (1). with.

and the fluctuating characteristic $N_0(x)=N_0=$ const. Such dependences approximately occur, for example, in the servo auto-selector when strobe pulses are considerably longer than signal ones.

Let us consider the case when feedback loop consists of one integrator: $K(p)=K/p$, but $\lambda(t)=\lambda_0$.

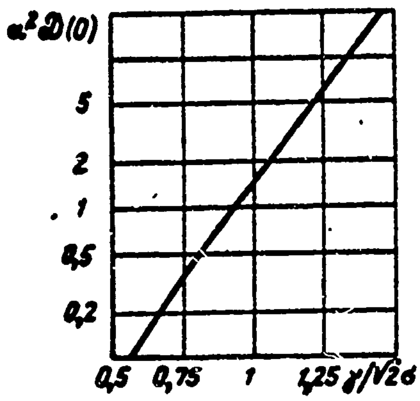

Fig. 5.8. Dispersion of the time milking of disruption/separation in first-order system.

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In this case the equation for the mean time takes form (5.30), where one should assume

$$B(x_0)=\frac{1}{2}K^2N_0,\ A(x_0)=-KF(x_0).$$

Using expression (5.36) of the general solution of boundary-value problem (5.30)-(5.31) taking into account the fact that

$$\varphi(x)=2\int\frac{A(x_0)}{B(x_0)}dx_0=-\frac{4A}{KN_0}|x_0|,$$

after some transformations we will obtain

$$m_1(x_0)=\frac{N_0}{4A^2}[a|x_0|+e^{a\gamma}-e^{a|x_0|}-a\gamma],$$

where

$$a=\frac{4A}{KN_0}.$$

In the particular case during zero initial disagreement/mismatch ($x_0=0$) the result takes the form

$$m_1(0)=\frac{N_0}{4A^2}(e^{a\gamma}-1-a\gamma).$$

3. Mean time to phase skip [jump-over] in system of phase automatic frequency control. The system of phase automatic frequency control whose functional diagram is depicted in Fig. 3.12, is described by the following differential equation:

$$\frac{d\varphi}{dt}=\Omega_n-K(p)[\Omega_y\sin\varphi-K\sqrt{N_0}\xi(t)], \qquad (5.58)$$

where $\varphi(t)$ - an instantaneous phase difference of the adjustable/tuneable generator and signal; K(p) - the operational gear ratio/transmission factor of filter; Ω_n— initial detuning of frequencies; Ω_y— band of retention; K - gear ratio/transmission factor of the element/cell, which manages frequency (reactance tube); N_0 - the spectral density of the white noise, led to the output of discriminator.

Let us consider the case when K(p)=n.

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Actually this occurs in the absence of filter (n=1) or during the use of the proportional-integrating filter $K(p)=\frac{1+pT_1}{1+pT}$ in the system

with the very large band of retention. in the latter case of $n=T_1/T$.

The determination of mean time to the first migration/jump of phase to the assigned magnitude for FAPCh systems was produced in works [17, 57, 77]. Let us consider the case when the synchronized generator is tuned to a frequency of received signal ($\Omega_H=0$). For determining the mean time to the first excess by the phase φ of values $\pm\varphi_1$ it is necessary to solve the boundary-value problem

$$\frac{K^2n^2N_0}{4}\frac{d^2m_1}{d\varphi_0^2}-n\Omega_y\sin\varphi_0\frac{dm_1}{d\varphi_0}+1=0, \qquad (5.59)$$

$$m_1(\varphi_1)=m_1(-\varphi_1)=0.$$

On the basis of general solution (5.36) let us register expression for the mean time m_1 during the zero initial disagreement/mismatch $\varphi_0=0$:

$$m_1(0)=\frac{4}{K^2n^2N_0}\int\limits_0^{\varphi_1}\int\limits_y^{\varphi_1}e^{a(\cos y-\cos x)}\,dx\,dy, \qquad (5.60)$$

where

$$a=\frac{4\Omega_y}{K^2nN_0}.$$

Using an expansion of integrands in the series/row of the Bessel functions, instead of (5.60) we will obtain [57]

$$m_1(0)=\frac{3}{K^2n^2N_0}\left[I_0^2(a)\varphi_1^2+4I_0(a)\sum_{k=1}^{\infty}\frac{I_{2k}(a)}{k}\times\right.$$

$$\left.\times\sin 2k\varphi_1+8\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(-1)^k I_m(a)I_k(a)\mathcal{J}(\varphi_1,\ m,\ k)\right], \qquad (5.61)$$

where

$$\mathcal{J}(\varphi_1,\ m,\ k)=\begin{cases}\frac{1}{m(k-m)}\cos(k-m)\varphi_1-\frac{4\cos k\varphi_1}{km}-\\ \quad -\frac{1}{k(k-m)} \text{ при } k\neq m,\\ \frac{1}{m^2}(1-\cos m\varphi_1) \text{ при } k=m.\end{cases}$$

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If it is necessary to determine mean time to the first migration/jump of phase on $\pm 2\pi$, then, setting in (5.61) $\varphi_1=2\pi$, we will obtain [17]

$$m_1(0)=\frac{2\pi^2}{i\Omega_y}aI_0^2(a). \qquad (5.62)$$

In a number of cases the transition of phase beyond the limits of $\pm\pi$ can be considered as the disruption/separation of tracking. Mean time before the onset of this event, as shown in [57], half the value, determined by expression (5.62).

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Chapter 6.

ANALYSIS OF THE DISRUPTION OF TRACKING WITH THE HELP OF ELECTRONIC COMPUTERS.

The use/application of analog and digital computer computational technology makes it possible investigated the complex problems cannot be analytically solved which at present. The methods of the study of the disruption/separation of tracking with the help of the digital and analog computers can be divided into two groups. Into the first group enter the methods for statistical testing, which make it possible to find the solution stochastic differential equations of servo system. The application of these methods in the analog and digital computers is stated in S 6.1, 6.2. Into another group enter the methods of the numerical solution of the equations of Fokker-Planck and Pontriagin, who describe the probabilistic characteristics of servo systems (S 6.3, 6.4).

6.1. Simulation of servo system in the analog computers.

The behavior of servo system, which is located under the action of the determined $\lambda_i(t)$ and random $\xi_i(t)$ disturbances, is described stochastic differential equation of the type

$$\frac{d^n x}{dt^n}=f\left(\frac{d^{n-1}x}{dt^{n-1}}, \frac{d^{n-2}x}{dt^{n-2}}, \ldots, x, \lambda_1(t), \lambda_2(t), \ldots, \xi_1(t), \xi_2(t)\ldots\right). \quad (6.1)$$

The solution of this equation is random function x(t), which characterizes change in the time of tracking error. It is obvious that by having sufficiently large group of the realizations of process x(t), by its corresponding working/treatment it is possible to obtain the necessary statistical characteristics of process, for example, the probability of disruption/separation for the preset time, the mean time to the disruption/separation, etc.

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This method of the definition of characteristics in the literature was called the method for statistical testing (Monte Carlo method) [6].

For the determination of the probability of the disruption/separation of tracking for time t_n is sufficient from total Mach number of the realizations of process x(t) to isolate

those N realizations in which the following error within time t_H exceeded the allowed values γ_1 or γ_2. Relation $N/M \approx P^*(t_H)$ gives the estimation of the probability of disrupting/separating the tracking. With increase of M the estimation asymptotically approaches a true value of the probability of disruption/separation. Thus, the task of determining the probability of disruption/separation is reduced to obtaining of the group of the realizations of process x(t) and its comparatively simple statistical processing.

One of the practical methods of obtaining the realizations x(t) is the simul.. on of differential equation (6.1) in the analog computer (AVM). Actually, gathering from the units of machine the necessary integrodifferentiating components/links and nonlinear devices/equipment and supplying the appropriate disturbances/perturbations, we will obtain the analog model whose behavior is described by equation (6.1). Observing the processes, which occur in the model, it is possible to judge the solution of equation (6.1). Are examined below only some special features/peculiarities of the construction of analog models for the solution of the problems about the disruption/separation of tracking. In more detail general/common/total questions of analog simulation are presented, for example, in monograph [9].

Construction of model. During the creation of analog model it is

convenient to proceed directly from the block diagram of the follower (see Fig. 1.2). With such method of simulation to each element/cell of block diagram is placed in the conformity its model, described by the same equations.

The overwhelming majority of the cascades/stages of servo system can be simulated/modelled on AVM with the help of the standard operational amplifiers, included by active and reactive/jet feedback. The models of the simplest linear components/links, which are frequently encountered in the regulating circuits, and their gear ratios/transmission factors are given in Table 6.1.

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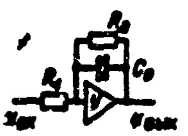
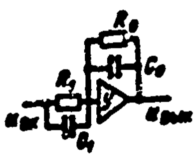
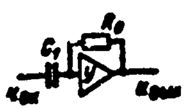
Table 6.1.

(1) Название каскада	(2) Связь между входом и выходом	(3) Схема модели	(4) Обозначения
(5) Безынерционный усилитель—инвертор	$u_{вых} = K u_{вх}$		$K=-\frac{R_0}{R_1}$
(6) Сумматор	$u_{вых} = K_1 u_{вх1} + K_2 u_{вх2} + \ldots + K_n u_{вхn}$		$K_i=-\frac{R_0}{R_i}$
(7) Интегратор	$u_{вых} = K\int_{t_0}^{t} u_{вх}\,dt$		$K=-\frac{1}{R_1C_0}$
(8) Инерционное звено	$u_{вых}=\frac{K}{1+pT}u_{вх}$ $p=\frac{d}{dt}$		$K=-\frac{R_0}{R_1}$ $T=R_0C_0$
(9) Пропорционально-интегрирующий фильтр	$u_{вых}=\frac{K(1+pT_1)}{1+pT}u_{вх}$		$K=-\frac{R_0}{R_1}$ $T=R_0C_0$ $T_1=R_1C_1$
(10) Дифференцирующее звено	$u_{вых}=K\frac{du_{вх}}{dt}$		$K=-R_0C_1$

Key: (1). Name of cascade/stage. (2). Connection/communication

between input and output. (3). Schematic of model. (4). Designations. (5). Inertia-free inverter-amplifier. (6). Summator. (7). Integrator. (8). Inertia component/link. (9). Proportional-integrating filter. (10). Differentiator.

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The parameters of the elementary cascades/stages, entering the model, are chosen in such a way that the coefficients of the differential equation of model would be proportional to the appropriate coefficients of initial equation (6.1).

Let us pause in more detail at the methodology of the construction of analog model. Let us consider the following specific problem.

Let us assume the analyzed regulating circuit (see Fig. 1.2) consists of the nonlinear inertialess discriminator with characteristic F(x), proportional-integrating filter $K_\phi(p)=\frac{1+pT_1}{1+pT}$ and control device with the operational gear ratio/transmission factor $K_y(p)=K/p$. At the input of discriminator functions regular disturbance/perturbation λ(t), at the output - broadband random process ξ(t) with a spectral density in the region of the lower frequencies of N_0. Stochastic differential equation, which describes

the behavior of the system in question, takes the form

$$T\frac{d^2x}{dt^2}+\left(1+KT\frac{dF(x)}{dx}\right)\frac{dx}{dt}+KF(x)=$$
$$=T\frac{d^2\lambda}{dt^2}+\frac{d\lambda}{dt}-K\xi(t)-KT_1\frac{d\xi}{dt}. \qquad (6.2)$$

Using Table 6.1, let us find for each element/cell of the block diagram (see Fig. 1.2) its analog model. Combining the models of separate units in accordance with the block diagram and introducing the necessary disturbances/perturbations in the form of stresses/voltages U_λ and U_ξ, we will obtain the common model of the ring of automatic control (Fig. 6.1). The designation/purpose of the separate network elements briefly is reduced to the following. The device/equipment, assembled on the operational amplifier У1, forms/shapes stress/voltage U_x, proportional to following error x(t). The unit of nonlinearity BN-1 reproduces the characteristic of discriminator F(x), undertaken with minus sign. Amplifier У2 performs the role of the summator, with the help of which into the diagram is introduced noise stress/voltage $\dot{U}_\xi(t)$. The proportional-integrating filter is assembled on the basis of operational amplifier У3, integrator - on the basis of amplifier У4. If the solved problem has not zero initial conditions, then into the amplifiers У3 and У4 must be introduced the corresponding stresses/voltages.

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So that the obtained diagram would be adequate to initial device/equipment, it is necessary to supply in congruence parameters of both systems. Let us introduce the scale factors, which connect stresses/voltages at the nodes of analog model with the processes in the reference system:

$$\begin{gathered} U_\lambda = M_\lambda \cdot \lambda(t), \quad U_x = M_x \cdot x(t), \\ \hat{U}_\lambda = \hat{M}_\lambda \cdot \hat{\lambda}(t), \quad U_F = M_F \cdot F(x), \\ U_\xi = M_\xi \cdot \xi(t), \quad U_z = M_z \cdot z(t). \end{gathered} \tag{6.3}$$

Furthermore, let us introduce concept of "machine" time $t_м$, connected with time t of differential equation (6.2) by the scale factor

$$M_t = \frac{t_м}{t}. \tag{6.4}$$

Time $t_м$ characterizes the reaction rate in the model. By the appropriate selection of coefficient M_t it is possible to ensure that processes in the model would proceed more rapid than real ones $(M_t<1)$ or slower $(M_t>1)$.

The junction/unit stresses/voltages of model (Fig. 6.1) are connected as follows:

$$\begin{gathered} U_x = \frac{R_{10}}{R_{11}} U_\lambda - \frac{R_{10}}{R_{12}} \hat{U}_\lambda, \quad U_v = \frac{R_{20}}{R_{21}} U_\xi + \frac{R_{20}}{R_{22}} U_F, \\ U_z = -\frac{R_{30}}{R_{31}} \frac{1 + C_{31} R_{31} p_м}{1 + C_{30} R_{30} p_м} U_v, \\ \hat{U}_\lambda = -\frac{1}{R_{41} C_{40}} \int_0^{t_м} U_z(\tau)\, d\tau = -\frac{U_z}{R_{41} C_{40} p_м}, \end{gathered} \tag{6.5}$$

where $p_м = d/dt_м$—differential operator in the machine time.

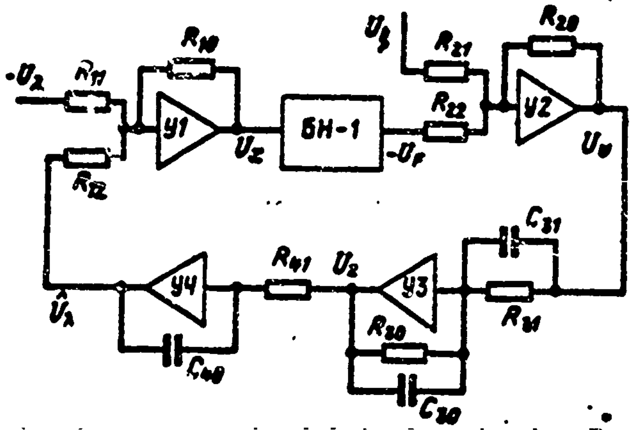

Fig. 6.1. Schematic of analog model.

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Let us combine the led dependences into one equation

$$U_x=\frac{R_{10}}{R_{11}}U_\lambda-\frac{R_{10}R_{30}}{R_{12}R_{31}R_{41}C_{40}}\frac{1}{p_M}\times$$
$$\times\frac{1+C_{31}R_{31}p_M}{1+C_{30}R_{30}p_M}\left[\frac{R_{20}}{R_{22}}U_F+\frac{R_{20}}{R_{21}}U_\xi\right]. \qquad (6.6)$$

Considering relationships (6.3)-(6.4), connecting machine and initial variable/alternating, let us represent equation (6.6) in the following form:

$$x(t)=\frac{R_{10}}{R_{11}}\frac{M_\lambda}{M_x}\lambda(t)-\frac{R_{10}R_{30}M_t}{R_{12}R_{31}R_{41}C_{40}M_x}\frac{1}{p}\times$$
$$\times\frac{1+\frac{C_{31}R_{31}}{M_t}p}{1+\frac{C_{30}R_{30}}{M_t}p}\left[\frac{R_{20}}{R_{22}}M_FF(x)+\frac{R_{20}}{R_{21}}M_\xi\xi(t)\right]. \qquad (6.7)$$

From identity condition of equations (6.2) and (6.7) we obtain,

that the parameters of analog model must satisfy the following requirements:

$$\frac{R_{10}M_{\lambda}}{R_{11}M_{u}}=1,\ \frac{R_{10}R_{20}R_{30}M_{t}M_{F}}{R_{12}R_{31}R_{41}C_{40}R_{22}M_{u}}=K,$$

$$\frac{R_{31}M_{F}}{R_{33}M_{\xi}}=1,\ C_{31}R_{31}=M_{t}\cdot T_{1},\ C_{30}R_{30}=M_{t}T. \tag{6.8}$$

As can be seen from (6.8), a number of coefficients, to be determined, exceeds a number of equations. This allows/assumes some arbitrariness in the selection of the parameters of analog circuit.

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The number of the operational amplifiers, entering the model, can be abbreviated/reduced, if we combine the operation of addition with the inertia conversions as this shown, for example, in Fig. 6.2. In this case input dynamic disturbance/perturbation is supplied into the model in the form of voltage $\dot{U}_{\lambda}$, process λ(t) proportional to derivative.

Noise stress/voltage $U_{\xi}(t)$, introduced into the analog model, must have the statistical characteristics, identical to initial process ξ(t). If machine time t_{M} differs from real t, then in accordance with the value of scale factor M_{t} should be corrected spectral noise density $U_{\xi}(t)$.

The accuracy of the determination of the probability of disruption/separation during the simulation on AVM is comparatively small and in essence it depends on the characteristics of the utilized machine (zero drift of operational amplifiers) and the stability of the generator of random stress/voltage. Usually in the standard universal computers it is possible to determine the threshold power of noise, with which the probability of disruption/separation does not exceed assigned value, with an accuracy to 10-20%. For the practice of this in the majority of the cases it is sufficient. If during the solution of problem is required high accuracy, then it is necessary to take special measures for the stabilization of noise source and to use machines with a small zero drift of operational amplifiers.

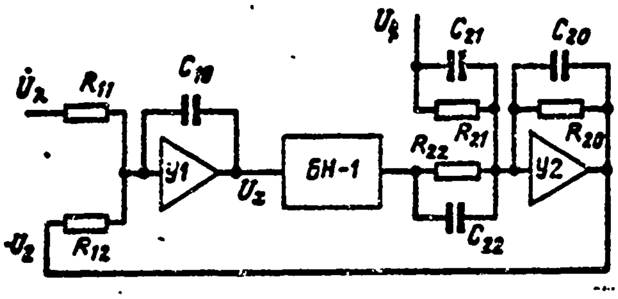

Fig. 6.2. The simplified circuit of model.

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6.2. Solution stochastic equations in the digital computers.

Similar to differential equations for the determined functions, stochastic differential equation can be solved by the means of discrete/digital computer technology. To questions of the use/application of electronic digital computers for solving the differential equations is dedicated a whole series of the books. As an example let us name monographs [3, 6]. Therefore in this work let us pause only at the short characteristic of the most important methods of solution and let us note the series/row of the special features/peculiarities, connected with the numerical solution stochastic equations.

The majority of the known methods of the numerical solution of ordinary differential equations can be spread also to the solution of

the equations, which contain random functions. As shown in S 6.1, for the definition of the characteristics of the disruption/separation of tracking it is necessary to develop the sufficiently large group of the realizations of the process being investigated. This can be done repeated solution of problem on TsVM.

The methods of solving the ordinary differential equations, including stochastic, can be divided into two classes:

1. Finite-difference methods, based on the series expansion of Taylor. They include [6] the methods of Euler, Runge-Kutta, Adams and the series/row of others.

2. Methods in which analyzed system of continuous action is substituted by equivalent discrete/digital system. The latter is described by equations in the finite differences which can be solved on TsVM. This class includes the methods of Boxer-Thaler, Bergen-Ragazzini, Tsypkin, etc. [3].

This classification is conditional, that as any method as a result is reduced to the solution of the finite-difference equations. In a number of cases the methods of the second class prove to be more economically; however, their practical use requires a comparatively great preparatory work on the composition of the algorithm of

solution for each specific problem. Furthermore, appear the difficulties of the analysis of nonlinear closed systems, which makes it necessary to artificially introduce into the feedback loop of the system being investigated delay line at least to one clock space of solution.

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The methods of the first class are more universal. Taking into account the noted special features/peculiarities, let us pause in greater detail at the methods of solving the differential equations, which relate to the first class.

Initial relationships/ratios. Let there be first-order ordinary differential equation

$$\frac{dx}{dt}=f(x, t),\ t>0,$$

and is assigned the initial condition $x(0)=x_0$. Let us find the solution of this equation on interval $0\leq t\leq t_n$ in a finite number of points $0<t_1<t_2<\ldots<t_i<\ldots<t_n$. For this we will use the expansion of function x(t) in the Taylor series in the vicinity of point t_i:

$$x(t)=x(t_i)+\frac{(t-t_i)}{1!}x'(t_i)+\frac{(t-t_i)^2}{2!}x''(t_i)+\ldots \quad (6.9)$$

Let us compute the first derivatives x(t) at point t_i:

$$x'(t_i) = [f(x, t)]\Big|_{\substack{x=x_i\\ t=t_i}},$$

$$x''(t_i) = \left[\frac{\partial f(x, t)}{\partial t} + \frac{\partial f(x, t)}{\partial x} f(x, t)\right]_{\substack{x=x_i\\ t=t_i}},$$

. . . .

Substituting into expression (6.9) and assuming/setting them $x(t) = x(t_{i+1})$, we will obtain recursion formula for calculating the values of function x(t) at the moments of time $t_1 < t_2 < \ldots$:

$$x(t_{i+1}) = x(t_i) + \frac{H}{1!} f(x(t_i), t_i) + \frac{H^2}{2!}\left[\frac{\partial f(x(t_i), t_i)}{\partial t_i} + \right.$$
$$\left. + f(x(t_i), t_i)\frac{\partial f(x(t_i), t_i)}{\partial x}\right] + \ldots, \qquad (6.10)$$

where $H = t_{i+1} - t_i$.

Retaining in this formula a sufficient number of terms, it is possible to compute the unknown function x(t) with the necessary accuracy. Depending on a quantity of terms of series/row (6.10), utilized for calculation $x(t_{i+1})$, they distinguish several methods of solution. The widest use received the following methods: Euler's method, which considers two members of series/row, by Euler-Cauchy (3 members) and Runge-Kutta (5 members of series/row and more).

Let us consider some of these methods in connection with the solution stochastic differential equations.

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Euler's method. As was shown in Chapter 2, stochastic differential equation of the n-order by the corresponding replacement of variable/alternating can be represented in the form of the system of nonlinear stochastic differential first-order equations. Let us register this system in the form

$$\frac{dx_k}{dt}=f_k(x_1, x_2, \ldots, x_n, t, \xi_1(t), \xi_2(t), \ldots, \xi_n(t)), \quad (6.11)$$

where $\xi_j(t)$ — the random functions of time, k=1, 2, ..., n.

For the uniformity of recording let us represent t in the form of variable/alternating $x_{n+1}=t$ and let us supplement to system (6.11) one additional differential equation

$$\frac{dx_{n+1}}{dt}=1.$$

As a result the reference system of equations can be registered in the form

$$\frac{dx_k}{dt}=f_k(x_1, x_2, \ldots, x_m), \; k=1, 2, \ldots, m, \quad (6.12)$$

where $f_m=1, m=n+1$.

Accordingly Euler's methods the value of functions at the end of (i+1) space $x_{k(i+1)}$ are found through the values of functions in the beginning of this step x_{ki} according to following formula [6]:

$$x_{k(i+1)}=x_{ki}+Hf_k(x_{1i}, x_{2i}, \ldots, x_{mi}). \quad (6.13)$$

where $H=t_{i+1}-t_i$ — step of solution.

Euler's method relates to the simplest methods of solving the differential equations. Its deficiency/lack is a comparatively low accuracy whose increase by decreasing the step H is not always possible due to the loss of stability of solution. Of this deficiency/lack is virtually deprived Runge-Kutta method.

Runge-Kutta method [6]. According to this method the solution of system (6.12) on (i+1) step is located through the values of functions $x_{1i}, x_{2i}, \ldots, x_{mi}$ at the previous space of integration for the formulas

$$x_{k(i+1)} = x_{ki} + \Delta x_{ki}, \quad k = 1, 2 \ldots, m,$$
$$\Delta x_{ki} = \frac{1}{6}(K_{k_1} + 2K_{k_2} + 2K_{k_3} + K_{k_4}),$$

moreover coefficients $K_{k_1}, K_{k_2}, K_{k_3}, K_{k_4}$ are determined by the expressions

$$K_{k_1} = Hf_k(x_{1i}, x_{2i}, \ldots, x_{mi}),$$
$$K_{k_2} = Hf_k\left(x_{1i} + \frac{K_{11}}{2}, x_{2i} + \frac{K_{21}}{2}, \ldots, x_{mi} + \frac{K_{m1}}{2}\right),$$
$$K_{k_3} = Hf_k\left(x_{1i} + \frac{K_{12}}{2}, x_{2i} + \frac{K_{22}}{2}, \ldots, x_{mi} + \frac{K_{m2}}{2}\right),$$
$$K_{k_4} = Hf_k(x_{1i} + K_{13}, x_{2i} + K_{23}, \ldots, x_{mi} + K_{m3}).$$

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In the reference system of equations (6.12) implicitly enter disturbances/perturbations $\xi_j(t)$, which are the random functions of

time. Entire temporary/time dependences in the digital computers are discrete/digital selections. Therefore during the simulation on TsVM of random processes it is necessary to manufacture this sequence of random numbers $\{\xi_i\}$, so that its statistical properties would be close to the properties of initial process $\xi(t)$.

Simulation of the uncorrelated sequences. Large role during the analysis of the disruption/separation of tracking play the weakly-correlated random processes. For the simulation of such processes on TsVM it suffices to manufacture the sequence of independent random numbers ξ_i, distributed according to required law $w(\xi_i)$.

It is usually assumed that $\xi(t)$ is gaussian process. However, if process $\xi(t)$ has broad band in comparison with the filter pass band in the feedback loop of system, then the one-dimensional law of distribution $w(\xi)$ does not play the significant role. Under these conditions independent of $w(\xi)$ the process is normalized by a filter. Therefore during the simulation it suffices as selections ξ_i to use the random numbers, distributed evenly. Since mathematical expectation of process $\xi(t)$ is usually equal to zero, random numbers ξ_i must be centralized. So that the sequence of numbers $\{\xi_i\}$ would be adequate to initial random process $\xi(t)$, it is necessary to ensure the equality of the spectral densities of initial process $\xi(t)$ and

the process, been simulated in the machine.

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For the elongation/extent of one step of solution value ξ_i, generated by machine, remains constant. The spectrum of the sequence of the square uncorrelated pulses with the random amplitudes and fixed/recorded durations H takes the form

$$N_\omega = \frac{4\sigma_\xi^2}{H\omega^2}(1-\cos\omega H), \qquad (6.14)$$

where σ_ξ^2 — dispersion of value ξ_i.

In the range of lower frequencies spectral density (6.14) is equal to

$$N_0 \doteq 2\sigma_\xi^2 H. \qquad (6.15)$$

The widest use in TsVM find random-number transducers χ_i, distributed evenly on interval [0, 1]. The mathematical expectation of the sequence of such numbers is equal to $m_\chi = 0{,}5$, and dispersion $\sigma_\chi^2 = 1/12$.

In order to simulate/model on the basis of numbers χ_i the uncorrelated central noise $\xi(t)$ with a spectral density of N_0 with the selected space of solution H, it suffices to use the following algorithm:

$$\xi_i = \sqrt{\frac{6N_0}{H}}(\chi_i - 0{,}5). \qquad (6.16)$$

Simulation of the correlated noise. With simulation of the correlated noise $\xi(t)$ with discrete/digital selection $\{\xi_i\}$ it is important to ensure the required law of distribution $w(\xi_i)$, and the required correlation function of numbers.

For the simulation of the sequence of numbers $\{\xi_i\}$, distributed according to the normal law, it is possible to use the central limit theorem of the probability theory, after taking as ξ_i the sum of independent random quantities χ_j, distributed according to the arbitrary law. Using a random-number transducer χ_j with the uniform law of distribution in interval [0, 1], it is possible to obtain numbers ξ_i, distributed according to the law, close to the normal, if we use the algorithm

$$\xi_i = \sigma_\xi \sqrt{\frac{12}{n}}\left(\sum_{j=1}^{n}\chi_j - \frac{n}{2}\right) + m_\xi.$$

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The sequence of numbers $\{\xi_i\}$, formed by algorithm (6.17), has a mathematical expectation m_ξ and a dispersion σ_ξ^2. In the majority of the practical cases for obtaining the normally distributed values ξ_i in expression (6.17) it suffices to sum up 5-7 random numbers χ_j.

The numbers, distributed according to the law, different from

the normal, can be obtained by the corresponding nonlinear conversion of initial numbers χ_i [3, 6]. In order to ensure the required law of the correlation of the developed numbers, can be used the method of sliding addition [3].

Calculation of the probability of disrupting/separating the tracking. Let us consider one of the possible methods of programming for determining the probability of disrupting/separating the tracking based on the example of regulating circuit (see Fig. 1.2) with the operational gear ratio/transmission factor of feedback loop

$$K(p)=\frac{K}{p(1+pT)(1+pT_1)}. \tag{6.18}$$

Let the discriminator be inertia-free nonlinear element/cell. with known discriminatory $F(x)$ and fluctuating $N_0(x)$ by characteristics (noise $\xi(t)$, converted to the output of discriminator, is broadband). At the entrance of system functions the dynamic disturbance/perturbation $\lambda(t)$, which is the known function of time. The initial state of system is assigned: $x(0)=x_{10}$, $\dot{x}(0)=x_{20}$, $\ddot{x}(0)=x_{30}$. By disruption/separation of tracking is understood the first output of following error $x(t)$ beyond the limits γ_1, γ_2 the aperture of discriminatory characteristic, moreover $\gamma_1<x_{10}<\gamma_2$.

On the basis (6.18) and the block diagram of the follower (see Fig. 1.2) let us compose stochastic differential equation relative to the current following error $x(t)$:

$$mT^2\frac{d^3x}{dt^3}+T(1+m)\frac{d^2x}{dt^2}+\frac{dx}{dt}+KF(x)=$$
$$=mT^2\frac{d^3\lambda}{dt^3}+T(1+m)\frac{d^2\lambda}{dt^2}+\frac{d\lambda}{dt}-K\xi(t),$$

where $m=T_1/T$.

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This equation can be represented in the form of system of equations of first-order:

$$\left.\begin{aligned}&\frac{dx_1}{dt}=x_2(t),\\&\frac{dx_2}{dt}=x_3(t),\\&\frac{dx_3}{dt}=-\frac{1+m}{mT}x_3(t)-\frac{1}{mT^2}x_2(t)-\frac{K}{mT^2}F(x_1)+\\&\quad+\Lambda(t)-\frac{K}{mT^2}\xi(t),\end{aligned}\right\}\qquad(6.19)$$

where

$$x_1(t)=x(t),\ \Lambda(t)=\frac{d^3\lambda}{dt^3}+\frac{(1+m)}{mT}\frac{d^2\lambda}{dt^2}+\frac{1}{mT^2}\frac{d\lambda}{dt}.$$

In order to determine one of the solutions stochastic system (6.19), we will use Euler's method. Let us decompose the time of observation t_n to n of the equal intervals H whose length let us take as the space of solution. In accordance with (6.13) the solution of system (6.19) at (i+1) step is determined by the values of variable/alternating at the previous i space:

$$\left.\begin{aligned}
&x_{1(i+1)}=x_{1i}+Hx_{2i},\\
&x_{2(i+1)}=x_{2i}+Hx_{3i},\\
&x_{3(i+1)}=x_{3i}-\frac{H}{T}\left(\frac{1+m}{m}x_{3i}+\frac{1}{mT}x_{2i}+\right.\\
&\qquad\left.+\frac{K}{mT}F(x_{1i})-T\Lambda_i+\frac{K}{mT}\xi_i\right),\\
&t_{i+1}=t_i+H,
\end{aligned}\right\}\quad(6.20)$$

where ξ_i — the random numbers, simulating effect $\xi(t)$ in accordance with algorithm (6.16), $\Lambda_i=\Lambda(t_i)$, $i=0, 1, 2, \ldots, n-1$.

In order to obtain M, solutions of system (6.19) and to calculate the probability of disruption/separation, it is possible to use the program, whose block diagram is depicted in Fig. 6.3. In the unit of initial data are introduced all constants, entering in (6.19). Mach numbers M and N are used for calculating the total number of realizations of process x(t) and number of realizations, in which occurred the disruption/separation of tracking.

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On the basis of the assigned initial conditions x_{1*}, x_{2*}, x_{3*} from formulas (6.20) consecutively/serially are computed functions $x_{1(i+1)}$, $x_{2(i+1)}$, $x_{3(i+1)}$. After each calculation $x_{1(i+1)}$ the result is equal with the boundary values γ_1 and γ_2. If $\gamma_1<x_{1(i+1)}<\gamma_2$, calculations are continued, otherwise is recorded the disruption/separation of

tracking, to number N is adjoined one and is produced transition to the calculation of new realization. The calculation of a total number of realizations is realized by an addition of one to Mach number after each turning to the initial conditions of task.

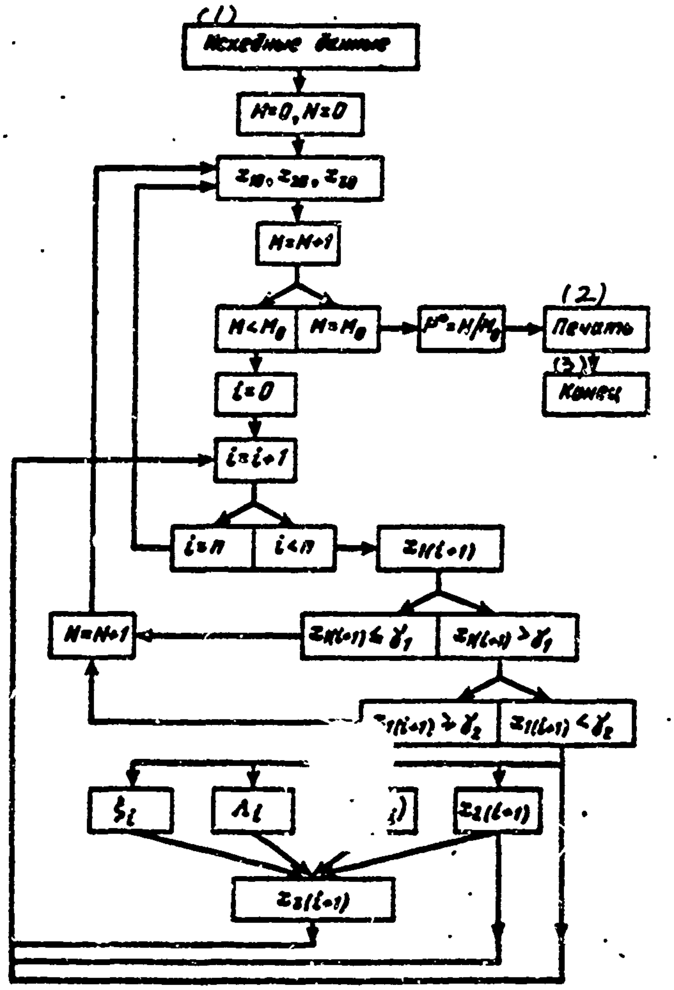

Fig. 6.3. Flowchart for determining the probability of disruption/separation.

Key: (1). Initial data. (2). Press/printing. (3). End/lead.

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When M reaches the given number M_t, the calculation of realizations

ceases and is counted the estimation of the probability of disrupting/separating the tracking $P^*(t_n) = N/M_0$. After the conclusion/output of the obtained result for the press/printing the process of solution is finished.

Analogously are composed programs also for calculating other statistical characteristics of disruption/separation.

The approximate estimate of number M_0, required for guaranteeing the assigned accuracy during the calculation of the probability of disruption/separation by the Monte Carlo method, can be obtained with the help of the asymptotic formula of De Moivre-Laplace:

$$\mathcal{P}\left(P-\Delta p_1<\frac{N}{M_0}<P+\Delta p_2\right)\approx$$
$$\approx\frac{1}{2}\left[\Phi\left(\sqrt{\frac{M_0}{2P(1-P)}}\Delta p_1\right)+\Phi\left(\sqrt{\frac{M_0}{2P(1-P)}}\Delta p_2\right)\right],$$

where $\mathcal{P}(\alpha<N/M_0<\beta)$ —probability that the frequency of disruptions/separations N/M_0, found from M_0 to realizations, lies/rests within the limits between α and β; $\Phi(x)$ - probability integral (1.5); P - probability of disruption/separation; Δp_1, Δp_2 - error in the determination of the probability of disruption/separation.

If the probability of disruption/separation P prior to the beginning of experiment is unknown even approximately, then with

certain supply in the accuracy it is possible to assume P=0.5.

Representation about the number of realizations M_{*}, required for determining the probability of disruption/separation with the assigned accuracy can be obtained from Fig. 6.4 which is constructed as follows.

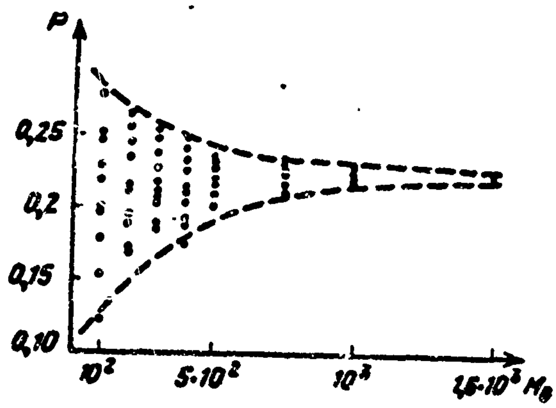

Fig. 6.4. To the convergence of the method for statistical testing.

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With selected number of starts M_0 (M_0=100, 200, ... 1500) was realized the series of 10 statistically independent trials on M_0 startings in each. In the course of each test was determined estimation $P^*=N/M_0$, which was noted in Fig. 6.4 by point. Produced experiment makes it possible to judge the spread of estimations with different sizes of samples M_0.

Selection of the space of solution. From the value of the space of discretization/digitization H in many respects depends the accuracy of the solution of problem. During the selection of space it is convenient to proceed from effective band width of the locked system

$$\Delta F_{э}=\frac{1}{2\pi K_x^2(0)}\int_0^\infty |K_x(j\omega)|^2 d\omega,$$

where $K_z(j\omega)$ — complex gear ratio/transmission factor of the linearized locked regulating circuit.

Usually it is impossible to analytically determine the required space of discretization/digitization, which would make it possible to find the probability of disruption/separation with the assigned accuracy. Therefore during the solution of such problems on TsVM selection space can be produced in the following manner. To determine the probability of disruption/separation with the selected initial space H_0, then the space to decrease 2-3 times and to again determine the probability of disruption/separation. If the obtained values of probabilities differ little, then H_0 is accepted for further solution. Otherwise the fragmentation of space is continued until probability is stabilized.

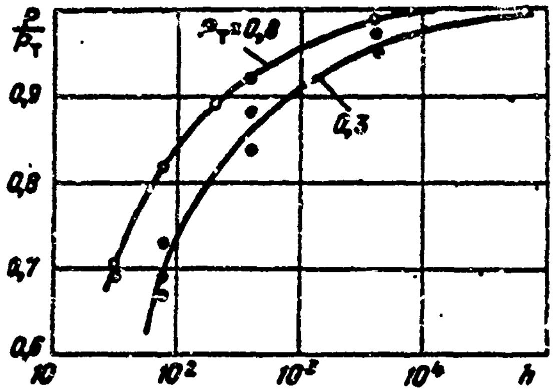

Fig. 6.5. Effect of the space of discretization/digitization on the accuracy of the determination of the probability of disruption/separation in first-order system.

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Let us give some considerations on the approximate selection of the initial space H_0. Use of relationship/ratio $H_0 = 1/2\Delta F_э$ of escape/ensuing from the theorem Kotelnikov, frequently gives inadmissibly the value of space H_0. The acceptable value H_0 strongly depends on that, we differentiate or not process x(t). In such a case, when process x(t) is not differentiated, the frequency of discretization/digitization must several orders exceed value $\Delta F_э$. As an example on Fig. 6.5 are constructed the calculated with the help of TsVM [ЦВМ - digital computer] graphs of the probability of disruption/separation in first-order servo system as the functions of dimensionless frequency $k = 1/2\pi H_0 \Delta F_э$. The values of probabilities P are calibrated with respect to precise values $P_т$. From the figure one can see that for determining the probability of disruption/separation with an accuracy to 10% should be taken the very low pitch:

$$H_0 \approx \frac{10^{-3}}{\Delta F_э}; \qquad (6.21)$$

If process x(t) is smooth as, for example, in the servo system of the second order with the integrating filter, then for obtaining

the same accuracy it suffices to take

$$H_0 \approx \frac{10^{-1}}{\Delta F_0}. \qquad (6.22)$$

For the system of tracking with the proportional-integrating filter K(p)=K(1+pT)/p(1+pT) the required space of discretization/digitization can be within the limits from (6.21) to (6.22) depending on value T_1/T.

6.3. Solution of equations in the partial derivatives in the analog computers.

Method of straight lines. At present for solving the boundary-value problems of mathematical physics were adopted simulator. Are known the examples when with their aid were solved equations in the partial derivatives with two, three and even four independent variables. The simplest simulators are grid models from the passive elements/cells for the solution of the problems of thermal conductivity.

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However, for the equations of Fokker-Planck or Pontriagin they are not applied due to the presence of members with first-order derivative for the space coordinates. In this paragraph is described

the method of straight lines (differential-difference method) for the solution of the problems about the disruption/separation of tracking in the analog computers. In principle the method of straight lines can be used, also, during the solution of the multidimensional equations, the practical difficulties of solving which are connected with the limited number of operational amplifiers in standard AVM.

Let us consider the use/application of a method of straight lines [12, 29, 82] for the solution on AVM of the one-dimensional equation of Pontriagin

$$\frac{\partial P(x,t)}{\partial t}=A(x)\frac{\partial P}{\partial x}+\frac{1}{2}B(x)\frac{\partial^2 P}{\partial x^2} \qquad (6.23)$$

with the boundary-value conditions

$$P(\gamma_1, t)=P(\gamma_2, t)=1, \qquad (6.24)$$

$$P(x, 0)=0, \ \gamma_1<x<\gamma_2 \qquad (6.25)$$

where P(x,t) - the probability of disrupting/separating the tracking for time t with the initial following error x.

Let us divide the segment $[\gamma_1, \gamma_2]$ into N intervals with a length of $\Delta x=(\gamma_2-\gamma_1)/N$ each. At the internal nodes

$$x_i=\gamma_1+i\Delta x \quad (i=1, 2, \ldots, N-1)$$

Let us replace derivatives by coordinate x with the finite-difference

analogs:

$$\left.\frac{\partial P(x,t)}{\partial x}\right|_{x_i}=\frac{P_{i+1}(t)-P_{i-1}(t)}{2\Delta x},$$

$$\left.\frac{\partial^2 P(x,t)}{\partial x^2}\right|_{x_i}=\frac{P_{i+1}(t)-2P_i(t)+P_{i-1}(t)}{(\Delta x)^2}, \tag{6.26}$$

where $P_i(t)=P(x_i, t)$.

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Substituting (6.26) in (6.23), we will obtain the system first-order of ordinary differential equations with the constant coefficients

$$\frac{dP_i(t)}{dt}=A_i\frac{P_{i+1}(t)-P_{i-1}(t)}{2\Delta x}+$$

$$+\frac{B_i}{2}\frac{P_{i+1}(t)-2P_i(t)+P_{i-1}(t)}{(\Delta x)^2}, \tag{6.27}$$

where $A_i=A(x_i)$, $B_i=B(x_i)$. Integrating (6.27) on the time, we will obtain the following system of equations:

$$P_i(t)=\int_0^t\left\{\left[\frac{B_i}{2(\Delta x)^2}-\frac{A_i}{2\Delta x}\right]P_{i-1}(\tau)-\frac{B_iP_i(\tau)}{(\Delta x)^2}+\right.$$

$$\left.+\left[\frac{B_i}{2(\Delta x)^2}+\frac{A_i}{2\Delta x}\right]P_{i+1}(\tau)\right\}d\tau. \tag{6.28}$$

System of equations (6.28) is gathered on the AVM with the help of N-1 integrators. Fig. 6.6 depicts the block diagram of obtaining solution in the i node/unit. To solution $P_i(t)$ in the i node/unit is placed into the conformity stress/voltage $u_i(t)$:

$$P_i(t)=Mu_i(t). \tag{6.29}$$

At end-points i=0 and i=N the solution is known: $p_0=p_N=1$. Therefore in these nodes/units are supported constant stresses $u_0=u_N=U$. Hence is determined the scale factor M=1/U. For the economy of a number of operational amplifiers in the diagram on Fig. 6.6 are not used the inverters between the nodes. Therefore stresses/voltages u_i at the nodes consecutively/serially change sign. In each node/unit stands the integrator with three entrances. Amplification factors in each entrance are equal to the appropriate coefficients of equation (6.28). Initial conditions are determined from expression (6.25)

$$u_i(0)=' \quad -1, 2, \ldots, N-1. \tag{6.30}$$

Sometimes it can seem tha. amplification factors for the set on AVM. In that case it is expedient to change the scale (see S 6.1).

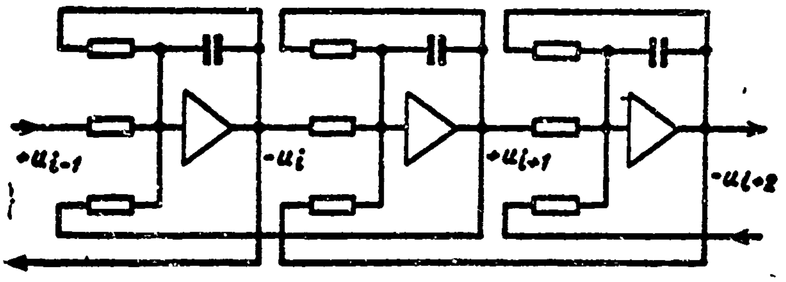

Fig. 6.6. Diagram of obtaining solution in the i node/unit.

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The accuracy of the solution of boundary-value problem (6.23)-(6.25) on AVM is determined by the systematic error in the method of straight lines and by the instrument error in the installation/setting up of the gear ratios/transmission factors of integrators.

The systematic error in the method of straight lines depends on the length of elementary interval Δx and error of the approximation of derivatives in equation (6.27). Previously we succeed in evaluating these errors only in simplest cases [29]. For equation (6.23) a priori estimations are not obtained.

We will be bounded to the determination of upper limit for value

Δx. Let us consider equation with the constant coefficients of removal/drift and diffusion

$$\frac{\partial P(x,t)}{\partial t}=A\frac{\partial P}{\partial x}+\frac{B}{2}\frac{\partial^2 P}{\partial x^2}. \qquad (6.31)$$

Without limiting generality, let us assume that $\gamma_1=0$, $\gamma_2=1$. The use/application of Fourier's method makes it possible to find in this case the exact solution of equation (6.31):

$$P(x,t)=1-\sum_{n=1}^{\infty}\frac{2[1-(-1)^n\exp(A/B)]}{\pi n+A^2/B^2\pi n}\times$$
$$\times\exp\left[-\frac{B}{2}\left(\pi^2 n^2+\frac{A^2}{B^2}\right)t\right]\exp\left(-\frac{A}{B}x\right)\sin(n\pi x), \quad (6.32)$$

with which it is convenient to be congruent/equate approximate solutions.

In order to obtain uniform boundary conditions, let us introduce the probability of retaining/preserving/maintaining the tracking $U(x,t)=1-P(x,t)$. The replacement of the variable/alternating $t_1=Bt/(\Delta x)^2$ converts the system of differential equations (6.27) for problem (6.31) to the following form:

$$\frac{dU_i(t_1)}{dt_1}=qU_{i-1}(t_1)-U_i(t_1)+pU_{i+1}(t_1), \qquad (6.33)$$

where

$$p=\frac{1}{2}+\frac{A}{2B}\Delta x,\ q=\frac{1}{2}-\frac{A}{2B}\Delta x. \qquad (6.34)$$

The eigenvalues λ of system of equations (6.33) are the roots of

the characteristic polynomial N-1 of degree. In order to determine them is comprised the system of recurrent relationships/ratios, which is solved by the method of z-conversions of Loran:

$$\lambda_i = -1 - 2\sqrt{pq}\cos\frac{i\pi}{N},\ i=1,2,\ldots,\ N-1. \quad (6.35)$$

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From (6.34) it follows that p+q=1. Therefore, if $p \geqslant 0$ and $q > 0$, then $p \leqslant 1$ and $q \leqslant 1$. Thus, $0 \leq pq \leq 1/4$. In this case from (6.35) it follows that all eigenvalues $\lambda_i < 0$. But if interval Δx is selected exaggerated, then either p<0 or q<0. As a result product pq<0 and eigenvalues λ_i become complex, which is impossible in the boundary-value problems for the equations of Fokker-Planck or first-order Pontriagin. Therefore during the use of a method of straight lines for solving equation (6.31) must be satisfied the condition

$$\Delta x < \frac{B}{|A|}. \quad (6.36)$$

In the general case, with the arbitrary coefficients of A(x) and B(x) in equation (6.23) it is impossible to find explicitly eigenvalues λ_i. Assuming that in the problems with the variable coefficients the appearance of instability of approximate solution carries local character, we consider that the condition of convergence (6.36) must be performed at each point cutting off

$\gamma_1 < x < \gamma_2$:

$$\Delta x < \min_{\gamma_1 < x < \gamma_2} \frac{B(x)}{|A(x)|}. \tag{6.37}$$

As basis/base for assumption (6.37) serves also proved for a series/row of the individual cases theorem [29]: for the convergence of the method of straight lines for certain equation it is sufficient c. . e ence of net point method for the same equation. As it will be shown into S 6.4, condition (6.37) is sufficient for the convergence of the method of walls. From this condition we find lower limit for a number of integrators, necessary during the solution of the boundary-value problem

$$N_{min} > \max_{\gamma_1 < x < \gamma_2} \frac{(\gamma_2 - \gamma_1)|A(x)|}{B(x)}. \tag{6.38}$$

During the practical use of a method of straight lines it is necessary to investigate its convergence by a consecutive increase in the number of divisions N of segment $[\gamma_1, \gamma_2]$. In Table 6.2 for case of A=2 and B=2 is illustrated the convergence of approximate solutions P(x,t) to precise (6.32) with an increase in the number of separations N (is accepted linear interpolation of the solutions between the nodes).

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The instrument error is connected with the fact that on AVM

inaccurately are displayed the factors of amplification of integrators. If in equation (6.28) all coefficients are put out absolutely accurately, then $\lim_{t\to\infty} P_i(t) = 1$. Inaccuracy leads to the fact that the sum of coefficients in equation (6.28) is not equal to zero; therefore steady-state solution is excellent from one.

The practical use of a method of straight lines showed that basic error is the systematic error, caused by a finite number of nodes N. With increase of N increases the weight of the instrument error.

Increase in the accuracy of the method of straight lines. In certain cases of the available number of integrators it can prove to be insufficiently for achievement of the required accuracy.

The first method of increasing the accuracy of solution lies in the fact that the points of the separation of segment $[\gamma_1, \gamma_2]$ distribute unevenly, congealing them in the region of maximum rate of change P(x,t) - near the boundary ones, points γ, γ_2.

Table 6.2.

x	t	P (x, t) при N (1)				
		2	4	6	10	(2) Точное
0,5	0,01	0,069	0,028	0,019	0,005	0,0009
	0,02	0,14	0,077	0,064	0,026	0,0276
	0,05	0,34	0,29	0,26	0,24	0,2482
	0,10	0,55	0,57	0,56	0,55	0,5604
	0,20	0,79	0,85	0,82	0,84	0,8517
	0,40	0,97	0,99	0,98	0,98	0,9831
0,25	0,01	0,52	0,081	0,11	0,077	0,0596
	0,02	0,58	0,20	0,18	0,16	0,1630
	0,05	0,67	0,39	0,41	0,38	0,3627
	0,10	0,77	0,62	0,63	0,61	0,6080
	0,20	0,89	0,88	0,85	0,86	0,8655
	0,40	0,98	1,00	0,97	0,98	0,9847

Key: (1). with. (2). Precise

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During the irregular-spaced separation of $h_{i,i-1}=x_i-x_{i-1}$ the derivatives on the coordinate are approximated as follows:

$$\left.\frac{\partial P}{\partial x}\right|_{x_i}=P_{i+1}\frac{h_{i,i-1}}{h_{i+1,i}(h_{i+1,i}+h_{i,i-1})}-P_i\frac{h_{i,i-1}-h_{i+1,i}}{h_{i+1,i}h_{i,i-1}}-$$

$$-P_{i-1}\frac{h_{i+1,i}}{h_{i,i-1}(h_{i+1,i}+h_{i,i-1})},\qquad (6.39)$$

$$\left.\frac{\partial^2 P}{\partial x^2}\right|_{x_i}=2\frac{P_{i+1}h_{i,i-1}-P_i(h_{i,i-1}+h_{i+1,i})+P_{i-1}h_{i+1,i}}{h_{i,i-1}h_{i+1,i}(h_{i+1,i}+h_{i,i-1})}.$$

The second method consists of a precise approximation of derivatives in comparison with (6.26). In the i node/unit during the calculation ∂P/∂x and ∂²P/∂x² we use values P_i not at three points, but in five:

$$\left.\frac{\partial P}{\partial x}\right|_{x_i}=\frac{-P_{i+2}+8P_{i+1}-8P_{i-1}+P_{i-2}}{12\Delta x},$$

$$\left.\frac{\partial^2 P}{\partial x^2}\right|_{x_i}=\frac{-P_{i+2}-P_{i-2}+16\,(P_{i+1}+P_{i-1})-30P_i}{12\,(\Delta x)^2}. \tag{6.40}$$

For the illustration of the advantage of the approximation of derivatives on five points (6.40) in comparison with the approximation on three points (6.26) let us consider the following example. The coefficient of removal/drift $A(x)=16xe^{-4x^2}$. This expression is a good approximation of the characteristic of frequency discriminator. In this case the maximums of characteristic are arranged/located at points x=±0.5, the absorbing boundaries are placed at points x=±1.5, where restoring force composes 5% of the maximum. The diffusion coefficient is placed equal to B=2. In Table 6.3 it is shown, as depends on a number of nodes/units N an absolute error in approximate solution (solutions are obtained at the moment of time t=4 at points x=1 and x=0).

Table 6.3.

Способ аппроксимации (1)	x	(2) Абсолютная погрешность приближенного решения при N				
		6	8	10	16	24
(6. 26)	1	0,607	0,133	0,081	0,029	0,015
	0	0,744	0,182	0,114	0,041	0,018
(6. 40)	1	0,093	−0,021	−0,014	−0,001	0,000
	0	0,037	−0,006	−0,002	−0,001	0,000

14*

Key: (1). Method of approximation. (2). Absolute error in approximate solution with N.

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As the exact solution is accepted the result, obtained with N=50: with x=1 P=0.491, with x=0 P=0.296.

The comparison of two methods of the approximation of derivatives (6.26) and (6.40) shows that during the use of five points is sufficient to have N=10-12. At the same time the use only of three points increases the necessary number of integrators N to 26-30.

In conclusion let us note that if the minimally necessary number of integrators N_{min}, found from (6.38), exceeds a number of integrators available in AVM or them is insufficient for achievement of the assigned accuracy, then it is necessary to use digital computers. For this system (6.27) of differential first-order equations is solved by the methods of linear algebra. Furthermore, the use/application of TsVM makes it possible to consider the case of time-varying of coefficients A(x,t) and B(x,t) of equation (6.23). In this case system (6.27) is converted into the system of differential equations with the variable coefficients. We arrive at our numerical solution by employing known finite-difference methods (method of Runge-Kutta, Adams, etc. [6]). Solution on AVM of equations with the time-varying coefficients to in practice carry out difficultly.

6.4. Solution of boundary-value problems in the digital computers.

The basic method of solution on TsVM of boundary-value problems for the equations in the partial derivatives is difference method [4]. The solution to the stationary equation for two-and three-dimensional problems is examined in work [32]. Boundary-value problems for the unsteady one-dimensional equations of Fokker-Planck are placed in the standard difference diagrams for the parabolic equation.

In this paragraph is examined the method of obtaining the explicit difference diagrams, based on the approximation of continuous Markov process with discrete/digital [90]. This method is

applied for the solution one- and two-dimensional unsteady problems. To solve the equations of higher order is difficult due to the existing limitations in the volume of working storage and operating speed of contemporary computers.

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One-dimensional problem. Let us consider the equation, which describes difference diagram for solving the one-dimensional equation of Fokker-Planck with the constant coefficient of diffusion B:

$$\frac{\partial w(x, t)}{\partial t}+\frac{\partial}{\partial x}[A(x) w]=\frac{B}{2} \frac{\partial^2 w}{\partial x^2}. \qquad (6.41)$$

Equations with the variable coefficient of B(x) are reduced to (6.41) with the help of the described into § 2.4 replacementof coordinate x.

Let us introduce the discrete model of the continuous Markov process r(t), examined/considered in the interval $\gamma_1 \leqslant x \leqslant \gamma_2$. We discretize many states of the Markov process x(t):

$$x_i = ih_x, \; i \in [l_1, l_2],$$

where

$$l_1 = \gamma_1/h_x, \; l_2 = \gamma_2/h_x.$$

It is assumed that the end-points γ_1 and γ_2 coincide with the nodes/units of discrete/digital Markov chain. At the moments of time $t_k = k\Delta t$ the Markov process x(t) under the action of noise disturbance/perturbation obtains the increase

$$\Delta x = x(t+\Delta t) - x(t) = \pm h_x.$$

So that the discrete/digital Markov process would converge to continuous, must be satisfied condition [18]

$$h_x = \sqrt{B}\sqrt{\Delta t}. \qquad (6.42)$$

The probability of increase $\Delta x = +h_x$ let us designate through p(x), and increase $\Delta x = -h_x$ — through q(x). The evolution of discrete/digital Markov chain is described by the equation of Markov

$$W(x, t+\Delta t) = p(x-h_x)W(x-h_x, t) + \\ + q(x+h_x)W(x+h_x, t), \qquad (6.43)$$

where $W(x, t) = w(x, t)h_x$ — probability of the stay in the node/unit with coordinate x at the moment of time t. For determining the probability W(x, t) equation (6.43) is written/recorded in the nodes/units of discrete/digital grid at the moments of time t_k:

$$W_l^{k+1} = p_{l-1}W_{l-1}^k + q_{l+1}W_{l+1}^k, \qquad (6.44)$$

where

$$W_l^k = W(lh_x, k\Delta t), \ p_l = p(lh_x), \ q_l = q(lh_x).$$

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The transition probabilities p and q are found from the condition for passage to the limit of difference equation (6.43) into the equation of Fokker-Planck (6.41) with $\Delta t \to 0$. Values p and q it is expedient to represent in the form

$$p(x)=\frac{1}{2}+C(x)h_x,\quad q(x)=\frac{1}{2}-C(x)h_x. \qquad (6.45)$$

Of both parts of equation (6.43) let us subtract W(x,t), let us divide on Δt and let us take into account relationships/ratios (6.42) and (6.45). As a result we will obtain the difference equation

$$\frac{W(x,t+\Delta t)-W(x,t)}{\Delta t}+\frac{2BC(x+h_x)W(x+h_x,t)-2BC(x-h_x)W(x-h_x,t)}{2h_x}=\frac{B}{2}\frac{W(x+h_x,t)-2W(x,t)+W(x-h_x,t)}{h_x^2}. \qquad (6.46)$$

From comparison (6.46) with (6.41) we find

$$C(x)=\frac{A(x)}{2B}. \qquad (6.47)$$

The obtained difference diagram is stable, if coefficients in equation (6.44) are non-negative [4]. This leads to the following condition:

$$\max_{\gamma_1<x<\gamma_2}\frac{|A(x)|}{B}h_x<1. \qquad (6.48)$$

The physical sense of (6.48) lies in the fact that the transition probabilities p and q satisfy conditions $0\leqslant p\leqslant 1$, $0\leqslant q\leqslant 1$.

The solution of boundary-value problem is reduced to the consecutive calculation of the probabilities of states W^k_l of discrete/digital Markov chain according (6.44) on each temporary/time layer $t_k=k\Delta t$ for all nodes $x_l=lh_x$, with exception of boundary ones, at which is assigned the condition for the absorption

$$W^k_{l_1}=W^k_{l_2}=0.$$

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At the moment of time $t=0$ is known initial distribution (2.40)

$$W^0_l=w_0(x_l)h_x,\ l\in[l_1+1,\ l_2-1].$$

The probability of disruption/separation $P(t_k)$ is located by the addition of the probabilities of the states Markov chain through the region $\gamma_1<x<\gamma_2$:

$$P(t_k)=1-\sum_{l=l_1+1}^{l_2-1}W^k_l. \qquad (6.49)$$

Example. Let us solve boundary-value problem for equation (6.41) with the linear coefficient of removal/drift A(x)=-Sx and the

symmetrical boundaries $\gamma_2=-\gamma_1=\gamma$. At zero time the following error is equal to zero: $w_0(x)=\delta(x)$. By the replacement of variable/alternating $x=2\gamma x_1$, $t=8\gamma^2 t_1/B$ equation (6.41) is reduced to the form

$$\frac{\partial w_1(x_1, t_1)}{\partial t_1} - a\frac{\partial}{\partial x_1}(x_1 w_1) = \frac{\partial^2 w_1}{\partial x_1^2},$$

where $a=8S\gamma^2/B$. Segment $-\gamma \leq x \leq \gamma$ is converted into the segment $-1/2 \leq x_1 \leq 1/2$. If the calculation of difference diagram (6.44) is begun directly from the temporary/time layer $t_1=\Delta t$, then as a result of extremely high rate of change in the solution near point $x=0$ with small t for achievement of a good accuracy it is necessary to take low pitches h.

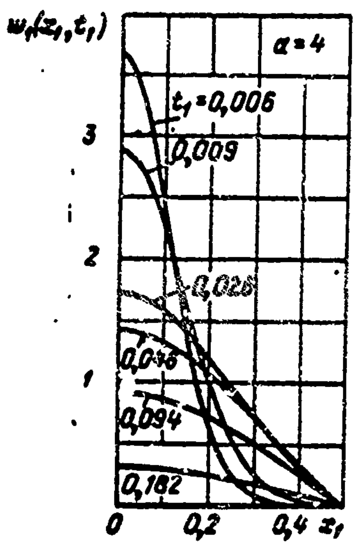

Fig. 6.7.

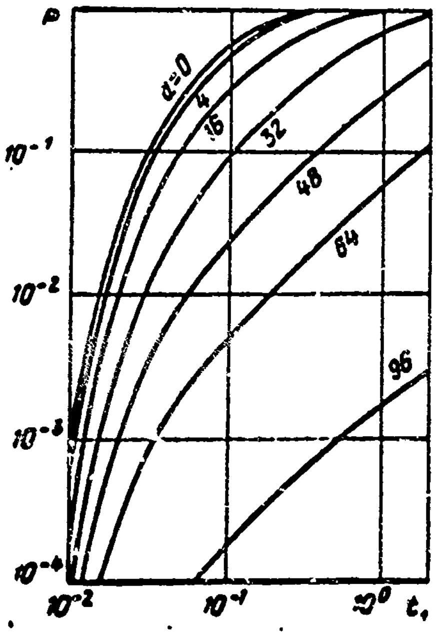

Fig. 6.8.

Fig. 6.7. Solutions of one-dimensional boundary-value problem for equation of Fokker-Planck.

Fig. 6.8 Probability of achieving boundaries in linear first-order system.

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On the other hand, with small t solution (2.44) of problem without taking into account boundary conditions does not manage considerably to spread. Therefore it is possible to find similar t' that with the

assigned error through the probability the representative point will be found in the segment $[-^1/_2, ^1/_2]$, and begin calculation from moment/torque $t=t'+\Delta t$. Fig. 6.7 shows dependences found thus of density of distribution $w_1(x_1, t_1)$ on coordinate x_1 at different moments of time. The probability of disruption/separation at the different values of parameter a is shown in Fig. 6.8. Since in the course of time rate of change of the solution $w(x, t)$ is decreased, then for the reduction in the volume of calculations it is expedient in resolving the boundary-value problem to enlargen space h_x and value Δt, connected with the space with relationship/ratio (6.42). In this case must not be broken condition (6.48).

Two-dimensional problem. The presentation of the methods of solving the two-dimensional boundary-value problems let us begin based on the example of control system with the integrator and the integrating filter in the feedback loop. As it follows from (2.71), the equation of Fokker-Planck in this case takes the form

$$\frac{\partial w(x, y, t)}{\partial t}+y\frac{\partial w}{\partial x}+\frac{\partial}{\partial y}[A(x, y)w]=\frac{B}{2}\frac{\partial^2 w}{\partial y^2}, \quad (6.50)$$

where

$$A(x, y)=-\frac{KF(x)-\lambda_1}{T}-\frac{y}{T}, \quad B=\frac{K^2N_0}{2T^2}.$$

For the convenience in (6.50) are introduced new in comparison with (2.70) the designations: $x=x_1$, $y=x_2$. Furthermore, it is accepted that the dynamic disturbance/perturbation is changed with a constant

velocity of $d\lambda/dt=\lambda_1$, but spectral density does not depend on the detuning: $N_0(x)=N_0$.

As shown in example 1 § 2.5, equation (6.50) is supplemented by the boundary conditions

$$w(x, y, t)\Big|_{\substack{x=\gamma_1\\0<y<\infty}} = w(x, y, t)\Big|_{\substack{x=\gamma_2\\-\infty<y<0}} = 0. \quad (6.51)$$

For obtaining the difference diagram let us introduce the discrete/digital two-dimensional Markov process, which approximates the continuous process $\{x(t), y(t)\}$. Let us decompose the region of tracking $\Omega(\gamma_1 \le x \le \gamma_2, -\infty<y<\infty)$ by the rectangular grid:

$$x_i = ih_x, \ i\in[I_1, I_2], \ I_1=\gamma_1/h_x, \ I_2=\gamma_2/h_x,$$
$$y_j = jh_y, \ j=0, \pm 1, \pm 2, \ldots$$

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Let us introduce discrete time $t_k=k\Delta t$, k=0, 1, 2, ... and as the two-dimensional discrete/digital Markov process let us take the following model. Since white noise $\xi(t)$ enters only into the second equation of the system stochastic equations (2.70), the at the moments of time t_k random abrupt bias/displacement endures only component y(t):

$$\Delta y = y(t+\Delta t) - y(t) = \pm h_y. \quad (6.52)$$

Component x(t) smoother function. In the interval of time Δt

according to the first equation of system (2.70) process x(t) obtains the increase

$$\Delta x = x(t+\Delta t) - x(t) = \int_t^{t+\Delta t} y(\tau)\,d\tau \approx y(t+\Delta t)\,\Delta t. \quad (6.53)$$

It is here accepted that in the interval between moments/torques t and t+Δt the value of component y is constant and equal to y(t+Δt). So that the digital process y_j would converge to continuous process of y(t) with Δt→0 just as in the one-dimensional problem, must be performed a specific ratio between values h_y and Δt:

$$h_y = \sqrt{B}\sqrt{\Delta t}. \quad (6.54)$$

The evolution of two-dimensional discrete/digital Markov chain is described by the following equation of Markov:

$$\begin{aligned} &W(x, y, t+\Delta t) = \\ &= p(x-\Delta x, y-h_y) W(x-\Delta x, y-h_y, t) + \\ &+ q(x-\Delta x, y+h_y) W(x-\Delta x, y+h_y, t), \quad (6.55) \end{aligned}$$

where W(x,y,t) $= w(x, y, t)h_x h_y$ — the probability of the stay in the node/unit with coordinates (x,y) at the moment of time t; p(x,y,) - the probability of increase $\Delta y = +h_y$; $q(x, y)$ — the probability of increase $\Delta y = -h_y$. During the numerical calculations equation (6.55) is written/recorded in the nodes of network with coordinates $x_i = ih_x$, $y_j = jh_y$ at the moments of time $t_k = k\Delta t$:

$$W_{i,j}^{k+1} = p_{i-1,j-1} W_{i-1,j-1}^{k} + q_{i-1,j+1} W_{i-1,j+1}^{k}, \quad (6.56)$$

where

$$W_{i,j}^{k} = W(ih_x, jh_y, k\Delta t),$$

$$p_{i,j} = p(ih_x, jh_y), \quad q_{i,j} = q(ih_x, jh_y).$$

During the composition of equation (6.56) it was considered that the representative point of discrete/digital Markov chain falls only into the mesh points. For this must be performed the following relationship/ratio between the increases in terms of all three variable/alternating

$$h_x = h_y \Delta t. \qquad (6.57)$$

The transition probabilities p and q are determined from the condition for the transition of difference equation (6.55) into the equation of Fokker-Planck (6.50) with $\Delta t \to 0$. For this we preliminarily convert the first two members in equation (6.50)

$$\frac{\partial w}{\partial t} + y\frac{\partial w}{\partial x} = \lim_{\Delta t \to 0} \frac{w(x, y, t + \Delta t) - w(x - \Delta x, y, t)}{\Delta t}, \qquad (6.58)$$

where $\Delta x = y\Delta t$ - increase in the direction, determined by the equation of integral curves in plane y=const:

$$\frac{dt}{1} = \frac{dx}{y}.$$

Substituting (6.58) in equation (6.50) and substituting derivatives on y by finite differences, we will obtain that the equation of Markov (6.55) with satisfaction of condition (6.54) passes in the equation of Fokker-Planck (6.50), if transition probabilities are determined by the equalities

$$p(x, y)=\frac{1}{2}+\frac{A(x, y)}{2B}h_y, \quad q(x, y)=\frac{1}{2}-\frac{A(x, y)}{2B}h_y. \tag{6.59}$$

Difference diagram (6.55) is stable with satisfaction of the condition

$$\max_{\substack{\gamma_1<x<\gamma_2 \\ -L<y<L}} \frac{|A(x, y)|}{B}<1. \tag{6.60}$$

With the numerical solution of problem for obtaining the finite number of nodes/units infinite with y region Ω it is necessary to

bound on certain level y=±L, which is admissible in connection with decrease w(x,y,t) with $y\to\infty$. On lines y=±L is assigned zero boundary condition.

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Thus, spaces on to the variable/alternating x, y and t are connected with three conditions: (6.54), (6.57) and (6.60). The solution of boundary-value problem consists of the calculation of the probabilities of the states of two-dimensional Markov chain according to formula (6.56) consecutively/serially for temporary/time layers $t_k=k\Delta t$.

Difference equation (6.56) is obtained from (6.55) on the assumption that the distance between the nodes along the axis x is equal to product $h_y\Delta t$ (6.57). In certain cases value h_x is so low that the volume of working storage TsVM proves to be insufficient for positioning/arranging the entire grid, considerably increases the count time. The use of larger/coarser space $h_x>h_y\Delta t$ leads to the fact that the representative point of discrete/digital Markov chain can not fall into the mesh points. Actually/really, if in difference equation (6.55) function W(x, y, t+Δt) is computed at nodes $x=ih_x$, $y=jh_y$, then function $W(x-y\Delta t, y\pm h_y, t)$ they are computed into points $(x-y\Delta t, y\pm h_y, t)$, not always coinciding with the mesh points.

The values of probability W at intermediate points are found out with the help of interpolation on x. The condition of convergence (6.60) of difference equation (6.55) does not depend on space h_x, value of which affects only the accuracy of interpolation.

Example. Let us consider the case of the linear discriminator: $A(x,y)=-ax-y$, $\gamma_2=-\gamma_1=0.5$ under the initial condition $w_0(x,y)=\delta(x)\delta(y)$.

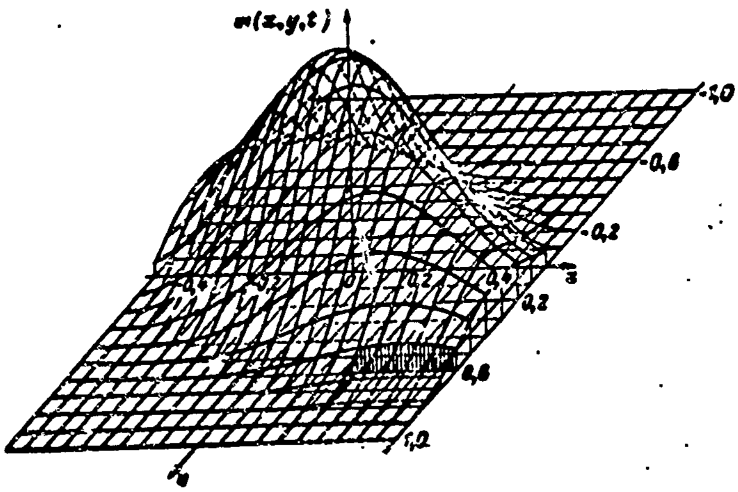

Fig. 6.9. Solution of two-dimensional boundary-value problem for the equation of Fokker-Planck.

Just as in the one-dimensional problem, the num cal solution conveniently to begin from certain moment/torque of time t', up to which the δ-function had time sufficiently to spread. Solution (2.44) of problem in the unlimited space at the moment of time t' is considered as the initial condition for the numerical calculation of difference diagram from the moment/torque of time $t'+\Delta t$. Fig. 6.9 shows distribution w(x,y,t) at the moment of time t=1.5, found for values of a=0.25, B=0.2. Solution is obtained when $h_y=0.1$, $h_x=0.005$, $\Delta t=0.05$, t'=0.45, L=1. From the figure one can see that on boundaries of x=±0.5, at points with a zero rate of y=0 distribution w(x,y,t) is

disruptive. Up to the moment/torque of time t=1.5 the probability of the absence of disruption/separation U(t)=0.930. It is determined by the addition of probabilities $W^{k}_{i,j}$ according to all nodes (i, j), which belong to region Ω.

For the confirmation of the correctness of the method of solving the boundary-value problem accepted for the two-dimensional equations of Fokker-Planck the obtained results were equal with the results of the digital simulation of the system stochastic equations (2.70) by the methods, presented into S 6.2. The comparison of the probabilities of disrupting/separating the tracking was conducted both in the linear ones and in the nonlinear control systems. In all cases is obtained a good coincidence of results.

Solution of the equation of Pontriagin. In this paragraph the probability of disruption/separation was determined indirectly - by t' . solution of boundary-value problem for the equation of Fokker-Planck with the subsequent integration of probability density for the region of tracking Ω. During the research of disruption/separation the probability distribution of following error is not usually of interest. Therefore to more expediently solve the equation of Pontriagin, since in this case is determined the dependence of the probability of disruption/separation $P(x_0, y_0, t)$ on the initial conditions $\{x_0, y_0\}$.

Let us consider based on the example of control system with the integrator and integrating filter (2.70) the solution of the equation of Pontriagin relative to the probability of achieving the boundaries of $P(x_0, y_0, t)$ for time t, if at the initial moment t=0 following error has components $\{x_0, y_0\}$:

$$\frac{\partial P(x_0, y_0, t)}{\partial t} = y_0 \frac{\partial P}{\partial x_0} + A(x_0, y_0) \frac{\partial P}{\partial y_0} + \frac{B}{2} \frac{\partial^2 P}{\partial y_0^2}. \quad (6.61)$$

Boundary conditions for equation (6.61) are obtained in the example, examined into S 2.6,

$$P(x_0, y_0, t)\big|_{(x_0, y_0)\in G^*} = 1, \quad (6.62)$$

where the regular part of boundary G^* form the straight lines $x_0 = \gamma_1$, $-\infty < y_0 < 0$ and $x_0 = \gamma_2$, $0 < y_0 < \infty$. Initial condition takes the form

$$P(x_0, y_0, t) = 0 \overset{(1)}{\text{при}} (x_0, y_0) \in \Omega - G^*. \quad (6.63)$$

Key: (1). with.

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At the moment of time t=0+0 at points $(\gamma_1, 0)$, $(\gamma_2, 0)$ the probability of disruption/separation is equal to

$$P(\gamma_{1(2)}, 0, 0+0) = \frac{1}{2}. \quad (6.64)$$

This is explained by the fact that at zero time rate y with probability P=1/2 can become either positive or negative. Therefore the representative point, which had at zero time of coordinate (γ_1, 0) or (γ_2, 0), at the subsequent moment of time will leave beyond the boundaries of the region Ω with probability P=1/2. Expression (6.64) should be taken as the initial condition for the nodes/units of difference diagram (γ_1, 0) and (γ_2, 0).

In order to use the difference diagram of the previous section, by the replacement

$$P(x_0, y_0, t) = 1 - e^{-t/T}\varphi(x_0, y_0, t) \qquad (6.65)$$

let us lead equation (6.61) to form (6.50)

$$\frac{\partial\varphi(x_0, y_0, t)}{\partial t} - y_0\frac{\partial\varphi}{\partial x_0} - \frac{\partial}{\partial y_0}[A(x_0, y_0)\varphi] = \frac{B}{2}\frac{\partial^2\varphi}{\partial y_0^2}. \qquad (6.66)$$

Equation (6.66) describes Markov process with the components $\{x_0, y_0\}$, which satisfy the unstable system stochastic equations

$$\left.\begin{aligned} \frac{dx_0}{dt} &= -y_0, \\ \frac{dy_0}{dt} &= \frac{KF(x_0) - \lambda_1}{T} + \frac{1}{T}y_0 + \frac{K}{T}\sqrt{N_0}\xi(t). \end{aligned}\right\} \qquad (6.67)$$

Approximating continuous Markov process $\{x_0, y_0\}$

discrete/digital, we will obtain the difference equation

$$\begin{aligned}\varphi(x_0, y_0, t+\Delta t) = \\ = p(x_0-\Delta x, y_0-h_y)\varphi(x_0-\Delta x, y_0-h_y, t)+ \\ +q(x_0-\Delta x, y_0+h_y)\varphi(x_0-\Delta x, y_0+h_y, t). \quad (6.58)\end{aligned}$$

In this case

$$\Delta y=\pm h_y, \Delta x=-y_0\Delta t,$$
$$h_x=h_y\Delta t, \ h_y=\sqrt{B\Delta t}.$$

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Difference equation (6.68) is stable with satisfaction of condition (6.60). Boundary conditions escape/ensue from comparison (6.65) with (6.62) and (6.63):

$$\varphi(x_0, y_0, t)\big|_{(x_0, y_0)\in G^*} = 0, \qquad (6.69)$$

$$\varphi(x_0, y_0, 0) = 1 \text{ при } (x_0, y_0) \in \Omega - G^*.$$

Key: (1). with.

For the proof of assumption (6.64) let us register difference equation (6.68) at the moment of time t=Δ[illegible] at point $x_0=\gamma_1$, $y_0=0$:

$$\varphi(\gamma_1, 0, \Delta t) = p(\gamma_1, -h_y)\,\varphi(\gamma_1, -h_y, 0) + {} + q(\gamma_1, h_y)\,\varphi(\gamma_1, h_y, 0).$$

Since according to (6.69) $\varphi(\gamma_1, -h_y, 0) = 0, \varphi(\gamma_1, h_y, 0) =$ [illegible] then, aperture q(x_0, y_0), we will obtain

$$\varphi(\gamma_1, 0, \Delta t) = \frac{1}{2} + \frac{A(\gamma_1, h_y)}{2B} h_y.$$

Passing to the limit with Δt→0, which involves $h_y \to 0$, let us find

$$\lim_{\Delta t \to 0} \varphi(\gamma_1, 0, \Delta t) = \frac{1}{2}.$$

The obtained result coincides with (6.64).

As the illustration Fig. 6.10 shows solution $P(x_0, y_0, t)$ at the moment of time $t=1.5$ for $A(x_0, y_0)=-0.25x_0-y_0$, $B=0.2$, $\gamma_2=-\gamma_1=0.5$. In contrast to the equation of Fokera-Planck during the solution of boundary-value problem for the equation of Pontriagin (6.61) does not succeed in finding the analytical solution, valid with small t. Therefore for achievement of identical accuracy it is necessary to take more fine pitches. Furthermore, region Ω is limited at the level of the high values L. All this causes an increase in the necessary volume of working storage of TsVM and count time. In given in Fig. 6.9 and 6.10 examples the time of solution of problem in the machine "BESM-4" is approximately/exemplarily 5-10 min.

The systems of the second order with the proportional filters. The method of solving the boundary-value problems presented can be used also for the analysis of the systems of control, in which in the feedback loop are correcting terms. In § 2.2 are described two methods of the introduction of multidimensional Markov process for such systems. During calculations on TsVM it is expedient to use the second method using which in the system stochastic equations (2.26) white noise enters only into one equation and in case (2.70) examined.

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In this case the equation of Fokker-Planck is simpler, which facilitates the composition of difference diagram.

As an example let us consider system with the integrator and proportional-integrating filter. The system stochastic equations (2.26) is analogous (2.70). Difference lies in the fact that the region of the trackings Ω in the phase space (z_1, z_2) is limited by the inclined lines

$$z_1+T_1z_2=\gamma_1, \quad z_1+T_1z_2=\gamma_2, \qquad (6.70)$$

which form boundary of G (see Fig. 2.3). The condition for absorption (2.75) is assigned therefore on the entire boundary of G of region Ω. As the illustration Fig. 6.11 shows solution $w(z_1, z_2, t)$ at the moment of time $t=1.5$ for the case $\lambda(t)=0$, $A(z_1, z_2)=-0.25(z_1+T_1z_2)-z_2$, $B=0.2$, $\gamma_2=-\gamma_1=0.5$, $n=T_1/T=0.5$.

Further observations. The solution of problems with the coefficient of diffusion B(x) depending on following error x leads to the nonuniform grids.

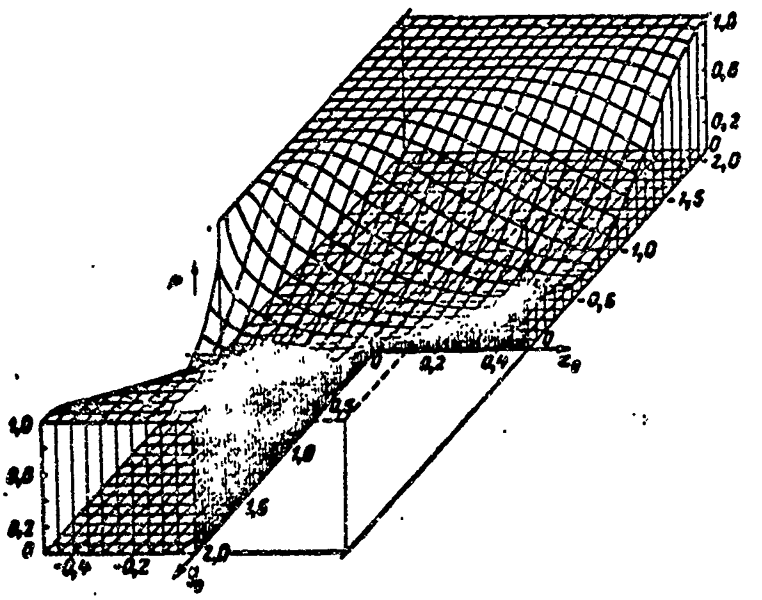

Fig. 6.10. Solution of two-dimensional boundary-value problem for the equation of Pontriagin.

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In some cases (see S 2.4) by the replacement of variable/alternating it is possible to give task to case of B(x)=const. But if the coefficient of diffusion B is constant, but the coefficient of removal/drift A(x, t) depends on time, then grid remains uniform, and the transition probabilities p and q become the functions of time. The latter fact virtually does not complicate the solution of problem.

It is in principle possible to compose difference diagrams, also, for the solution of three-dimensional unsteady problems. However, the existing limitations in the volume of the working storage of contemporary TsVM considerably narrow parametric domain in which can be solved three-dimensional task.

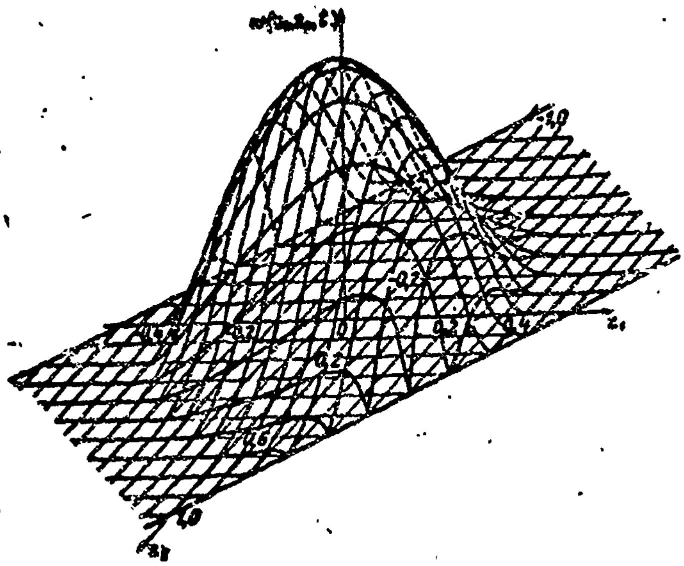

Fig. 6.11. Solution of two-dimensional boundary-value problem for the system with proportional-integrating filter.

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CONCLUSION.

In the present monograph is examined the series/row of the methods of the analysis of the disruption/separation of tracking, most frequently used in the practice. This makes it possible to do some conclusions about the possibilities of one or the other method and advisability of its use/application under the specific conditions.

The greatest possibilities for the analysis of the disruption/separation of tracking possesses the method for statistical testing (Monte Carlo method). With its aid it is possible to determine the characteristics of disruption/separation for the very broad class of regulating circuits. In this case the mathematical model of system can be constructed taking into account many thin special features/peculiarities of the work of concrete/specific/actual device/equipment.

An essential deficiency/lack in the method for statistical testing is the need of applying the computers. This raises in price research and does not give the possibility to obtain analytical dependences.

The Monte Carlo method sufficiently successfully is realized both on the analog ones and in the digital computers. The latter, however, ensure the considerably high accuracy of the obtained results. A method for statistical testing it is difficult to use in cases when it is necessary to investigate the work of regulating circuit during the long time of observation. It is inconvenient also for the analysis of systems with the very small probabilities of disrupting/separating the tracking $P \leq 10^{-3}-10^{-3}$, since in this case appears the need for carrying out a large number of launchings/startings of machine.

Among the analytical methods the greatest accuracy possess the methods, which are based on the theory of Markov processes. Unfortunately, their use/application is significantly limited to the order of the analyzed system. Most successfully they are used for the analysis of disruption/separation in first-order systems. In this case for the fixed systems it is expedient to apply the method, which is based on the relationship/ratio of Kramers, for the time-dependent systems - Bubnov-Galerkin method or the method of the compensating sources. With the complication of the conditions for the work of system increases the labor expense for the solution of problem.

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Thus, if Kramers process gives sufficiently simple calculated correlations, then Bubnov-Galerkin method leads to the simple results only in the case of the sufficiently high level of the noise effect when for determining the probability of disruption/separation it suffices to be bounded to the first or second approximation/approach. On a small noise level it is necessary to seek higher approximations/approaches, that it is possible to do with the help of electronic computational engineering. In a number of cases Bubnov-Galerkin method successfully is combined with the asymptotic

method which makes it possible sufficient simply to determine the high eigenvalues of solution.

The method of the compensating sources makes it possible to find the probability of disruption/separation in the systems, subjected to complicated dynamic effects; however, it requires comparatively labor-consuming calculations of definite integrals. In cases when discriminatory characteristic can be approximated by the piecewise-linear dependence with a small number of salient points (on the order of two-three), all calculations can be carried out by hand. With the complicated characteristics for the calculation it is necessary to use a computer. In these cases the method of the compensating sources usually loses its advantages in comparison with the method for statistical testing. Furthermore, one should also consider that the latter/last method requires smaller preparatory work in constructing a program of solution.

For the analysis of the systems of the second order to apply the theory of Markov processes somewhat more difficult. The sufficiently well analytical methods of solving the equations of Fokker-Planck are developed only for the stationary regulating circuits. However, in these cases it is possible to determine the probability of disruption/separation not in any parameters of servo system. Successfully yield to analysis systems with the high or small fading.

In the intermediate cases it is necessary to introduce in the calculated relationships/ratios of correction in the form of the coefficients, determined experimentally (see S 3.2).

The proximate analysis of the time-dependent systems of the second order can be carried out with the help of the method of the compensating sources. Errors in this method substantially increase in comparison with the analysis of first-order systems.

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The statistical characteristics of the servo systems of the first and second orders with the very high accuracy can be calculated by the method of solution of the corresponding equations of Fokker-Planck and Pontriagin on the electronic computers.

For the analysis of systems of higher than the second order to apply the theory of Markov processes is very difficult and at present this is virtually not done.

By nature and formulation of the problem to the analysis of disruption/separation are close the questions, decided in the theory of ejections. Therefore sometimes for the analysis of the disruption/separation of tracking it is possible to use the results,

obtained in the theory of ejections. Most successfully this can be done when the system of tracking in the limits of the aperture of discriminator is considered linear, and following error is the differentiated random function of time. The order of servo system in this case does not play the significant role.

Connecting the analysis of the disruption/separation of tracking with the theory of ejections, it must be noted that many questions of the theory of ejections comparatively easily are solved, if is determined the probability of the first reaching/achievement of threshold level. So, if for the stationary random process of x(t) is known probability that x(t) in the time interval of observation at least one time will leave for the level γ, then by simple calculations it is possible to find, in particular, such characteristics of the ejections of process x(t) above the level γ as the frequency of ejections, the distribution of the durations of ejections and intervals between them, the distribution of the greatest values, attained by process of x(t) in the time interval of observation and the like [33].

The approximate estimate of the quality of the work of servo systems under the conditions for noise effect they can give also the characteristics, less complete than the probability of disruption/separation for the preset time of observation. They

include, for example, mean time and the dispersion of time to the disruption/separation, the critical power of noise at which the disruption/separation it is possible to virtually yet not be considered the like. The time characteristics of disruption/separation (mean time, dispersion) with the sufficiently high accuracy comparatively simply are determined for first-order servo systems. For the systems of higher order their calculation is connected with the solution of partial differential equations. The critical power of noise is determined comparatively simply; however, it characterizes the phenomenon of disruption/separation very approximately.

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No typing.